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THEORY OF MICROPOLAR ELASTICITY

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Princeton, New Jersey

June 1967

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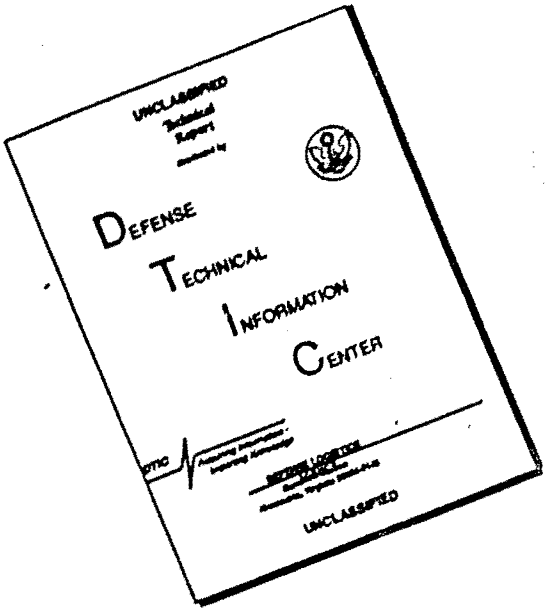

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THEORY OF MICROPOLAR ELASTICITY

by A. Cemal Eringen

for Office of Naval Research
Department of the Navy
N00014-
Contract N-0014-67-A-0151-0004

Technical Report No. 1
June 1967

PRINCETON UNIVERSITY
Department of Aerospace and Mechanical Sciences

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ABSTRACT

In this article we present a self-contained account of the recent theory of micropolar elasticity. Micropolar elastic materials, roughly speaking, are the classical elastic materials with extra independent degrees of freedom for the local rotations. These materials respond to spin inertia and body and surface couples, and as a consequence they exhibit certain new static and dynamic effects, e.g., new types of waves and couple stresses. The theory is fully deterministic as against the background of the recently popular indeterministic couple stress theory (cf. Art. 23). The mechanics of certain classes of materials with fiberous and elongated grains (e.g., dumbbell types of grains) represents a potential field of application of the theory.

The geometry of deformation and its measures are introduced on a more general background of materials exhibiting granular and microstructure effects (micromorphic materials). Various types of microstrains and microrotations are discussed. Compatibility conditions for the micropolar strains are derived. The kinematics of strains, microstrains and rotations are presented. Basic laws of motion, conservation of mass, conservation of microinertia, balance of momentum, balance of moment of momentum, conservation of energy are postulated and their local forms are obtained. The thermodynamics of micropolar solids is formulated and the consequences of the entropy inequality are discussed. Constitutive equations are found for the linear theory of micropolar elasticity. The basic field equations and initial and boundary conditions are given.

The indeterminate couple stress theory is shown to result as a special case of the theory when the motion is constrained. Several static and dynamic problems are solved to reveal some of the new physical phenomena exhibited by

the theory. These include propagation of waves in infinite micropolar elastic solids, reflection of various types of micropolar waves in a half space, surface waves, the stress concentration around a circular hole in a tension field, and force and moment singularities in an infinite solid. The Papkovitch and Galerkin representations are presented.

The article is based mostly on the works of Eringen and his co-workers published during the past several years. Many parts, however, contain new compositions and several other results are presented for the first time.

1. INTRODUCTION

Classical continuum mechanics is based upon the fundamental idea that all material bodies possess continuous mass densities, and that the laws of motion and the axioms of constitution are valid for every part of the body no matter how small they may be. Accordingly, a small volume ΔV enclosed within a surface ΔS possesses a mass density ρ defined by

$$\rho \equiv \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} \tag{1.1}$$

where Δm is the total mass contained in ΔV. Here ρ is independent of the size of ΔV and depends only on the position vector $\underset{\sim}{x}$ of a point in ΔV and time t. Consider the following experiment for the measurement of ρ: The mass density of a homogeneous material may be calculated approximately by weighing a large number of pieces having different volumes and calculating the ratio $\Delta m/\Delta V$ for each piece. If the resulting numbers ρ are plotted against ΔV, one finds that this ratio is nearly constant when ΔV is greater than certain critical volume ΔV^* and begins to show dependence on ΔV when $\Delta V < \Delta V^*$. The size of ΔV^* depends on the constitution of the material. As ΔV approaches zero, this dependence becomes violent, Fig. 1.1. This situation is well-understood when we remember the granular and molecular nature of materials. The classical continuum theory may therefore not represent a good mathematical model for the approximation of a physical theory in the range $\Delta V < \Delta V^*$.

The loss of accuracy in classical continuum mechanics may stem from another important reason. If the response of the body is sought to an external physical effect in which the length scale is comparable to the average grain or molecular size contained in the body, the granular or molecular constituents of the body are exited individually. In this case, the intrinsic motions of the constituents (microelements) must be taken into account. This point becomes clear especially in connection with the propagation of waves having high frequencies or short wave lengths. When the wave length is of the same order of magnitude as the average dimension of the microelements, the intrinsic motions of the microelements of ΔV with respect to the center of mass of ΔV can affect the outcome appreciably. This situation previals in practical applications when the material under consideration is a composite material containing macromolecules, fibers, and grains. For such materials, the critical volume ΔV^* is of the order of magnitude of the cube of a fraction of an inch to several inches. Solid propellant grains, polymeric materials, and fiberglas are but a few examples for such materials.

Another example is the anomalous behavior of blood when flowing through capillaries. Blood consists of a fluid (plasma) in which are dispersed elements of microscopic dimensions (corpuscles). When blood flows through capillaries whose diameter is comparable to that of the corpuscles, the flow characteristics differ from those for large vessels. As a further example, experiments have shown that the resistance of a solid to surrounding fluid can be reduced by as much as 1/3 when a minute amount of additives are cast into fluid.

It may be conjectured that a rational treatment of surface tension, microcracks, microfracture, and the mechanics of granular media and composite materials ultimately will have to be based on the theory of microcontinua. The nature of experimental work on the properties of such materials, no doubt, will be affected with these developments.

Presently there exist several approaches to the formulation of micromechanics. Some of these theories are very general in nature but incomplete and not closed. Others are concerned either with special types of material structure and/or deformations. Fundamental ideas contained in some of these theories can be traced all the way to Bernoulli and Euler in connection with their work on beam theories. In the elementary beam theory, with each section of the bar there is associated two sets of kinematical quantities, namely, a deformation vector and a rotation vector and two types of internal loads, namely, the tractions and couples. In plate theory, we have a similar situation. Bar and plate theories involving these independent quantities were recorded by Kelvin and Tait[1879]. The existence of the stress couples independent of tractions is essential to these theories. For three-dimensional bodies, this concept is found in the work of MacCullagh [1839] in connection with his work on optics. Lord Kelvin went as far as building a model of what he called "quasi-rigid" ether which is supposed to provide a mechanical model for Maxwell's theory of electromagnetism. The existence and basis of couple stress in elasticity was also noted by Voigt [1887] in connection with polar molecules.

In a remarkable monograph, E. and F. Cosserat [1909] gave a unified theory for the deformable bars, surfaces, and bodies. A Cosserat continuum is defined as a three-dimensional continuum, each point of which is supplied with a triad[1]. By the use of a principle which they call "euclidean action"[2] and by calculating the variation of the internal energy density, they gave the equations of local balance of momenta for stress and couple stress and the expressions of surface tractions and couples. In the work of the Cosserat brothers, we find that the effect of couple stress on the motion of deformable bodies is fully taken into account.

Some fifty years elapsed after the work of the Cosserats with very little activity in this field. The idea of a Cosserat continuum was revived in various

---

[1] In the terminology of Truesdell and Noll [1960, Art. 256] "directors", the same terminology and similar ideas were used by Toupin [1964], Green, Rivlin and Naghdi [1965].

[2] Equivalent to the principle of objectivity, cf. Eringen [1962, Art. 27].

special forms by Günther [1958], Grioli [1960], Aero and Kuvshinskii [1961], and Schäfer [1962] of whom Günther also remarked on the connection to the theory of dislocations. The question of couple stress was reopened with an incomplete theory of Cosserat bars and surfaces included in Truesdell and Toupin [1960]. Mindlin and Tiersten [1962], Toupin [1962], Eringen [1962, Arts. 32, 40] recapitulated a special Cosserat continuum now known as the (indeterminate) couple stress theory. In these theories, the rotation vector is not an independent vector, consequently the antisymmetric part of the stress and symmetric part of the couple stress remains undetermined (cf. Art. 23 below). Eringen and Suhubi [1964a,b] and Eringen [1964] introduced a general theory of a nonlinear microelastic continuum in which the balance laws of continuum mechanics are supplemented with additional ones, and the intrinsic motions of the microelement contained in macrovolume ΔV are taken into account. This theory, in special cases, contains the Cosserat continuum and the indeterminate couple stress theory. Independently, a microstructure theory of elasticity was published by Mindlin [1964] and a multipolar continuum theory by Green and Rivlin [1964]. Both of these theories appear to have contacts with those of Eringen and Suhubi, in special situations. Following these works, an intense activity began and literature now contains several hundred papers in this and in related fields. A proper assessment of these works with appropriate references is beyond the scope of this article.

This article is concerned, basicially, with special types of continua called <u>micropolar continua</u>. The theory was initiated by Eringen and Suhubi [1964b, Art. 6] as a special case of their work on the microelastic solid and was named couple stress theory. Later, Eringen [1966a,b] recapitulated and renamed it micropolar theory and proved several uniqueness theorems. A similar theory appears to be given, independently, by Palmov [1964] for the linear elastic solid. While the theory is fresh and no experimental work has been published as yet, we believe that the results obtained so far are sufficient to strengthen the future of the theory.

The theory of micropolar elasticity is concerned with material media whose constituents are dumbbell molecules. These elements are allowed to rotate independently without stretch. The theory is expected to find applications in the treatment of mechanics of granular materials with elongated rigid grains and composite fiberous materials.

The first eight sections (Arts. 2 - 9) of this article give a treatment of the geometry of deformation and microdeformation, strain and rotation measures, compatibility conditions, and some illustrative examples of deformation. Sections 10 - 13 are devoted to kinematics and rate measures. External and internal loads and the balance laws are discussed in Sections 14 to 17, and energy and entropy in Sections 18 and 19. The constitutive equations of the theory of micropolar elasticity and restrictions on the coefficients are derived in Sections 20 and 21. The field equations, boundary and initial conditions are prescribed and discussed in Section 22. In Section 23, we show how the indeterminate couple stress theory arises as a special case of the micropolar theory. Sections 24 through 29 are devoted to the solutions of various problems.

Micropolar continuum mechanics is in the stage of its infancy. The linear theory is reasonably simple and it lends itself to the solution of some important boundary and initial value problems. A large class of unsolved problems and experimental work offer a challenge to future workers.

2. DEFORMATION AND MICRODEFORMATION

A material point P of a body B having volume V and surface S in its undeformed and unstressed state may be located by its rectangular coordinates X_1, X_2, and X_3 (or simply X_K, $K = 1,2,3$), Fig. 2.1. If the body is allowed to move and deform under some external loads, it will occupy a region having volume $\mathcal{V}$ and having surface $\mathcal{S}$. Referred to the same rectangular frame of reference, the new position of the point P will be x_1, x_2, x_3 (or simply x_k, $k = 1,2,3$). Under the assumption of <u>indestructibility</u> and <u>impenetrability</u> of matter, each material point in the undeformed body B will occupy a unique position in the deformed body $\mathcal{B}$.

Conversely, each point in $\mathcal{B}$ can be traced back to a unique point in B. Thus, the deformation of the body at time t may be prescribed by a one-to-one mapping

$$x_k = x_k(X_1,X_2,X_3,t), \qquad k = 1,2,3 \tag{2.1}$$

or its inverse motion

$$X_K = X_K(x_1,x_2,x_3,t), \qquad K = 1,2,3 \tag{2.2}$$

We assume that (2.2) is a unique inverse of (2.1) for all points contained in the body except possibly some singular surfaces, lines, and points. For this to be valid, the three functions $x_k(X_1,X_2,X_3,t)$ must possess continuous partial derivatives with respect to X_1, X_2, and X_3 for all times, and the jacobian

$$J \equiv \det \frac{\partial x_k}{\partial X_K} = \begin{vmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{vmatrix} \tag{2.3}$$

must not vanish. Henceforth, we assume this to be the case.

The partial derivatives

(2.4) $$x_{k,K} \equiv \frac{\partial x_k}{\partial X_K} \quad , \quad X_{K,k} \equiv \frac{\partial X_K}{\partial x_k}$$

are called deformation gradients, and they are basic in the study of continuum mechanics.

We now consider a volume element ΔV enclosed within its surface ΔS in the undeformed body. Let the center of mass of ΔV have the position vector $\underset{\sim}{X}$. All materials possess certain granular and fibrous structure with different sizes and shapes. If the physical phenomena under study has a certain characteristic length (such as wave length) that compares with the size of grains in the body, then the microstructure of the material must be taken into consideration. In such situations, classical continuum mechanics must be modified by considering the effect of the granular character of the medium. Suppose that the element $\Delta V + \Delta S$ contains N discrete micromaterial elements $\Delta V^{(\alpha)} + \Delta S^{(\alpha)}$, $(\alpha = 1,2,\ldots N)$, each with a mass density $\rho^{(\alpha)}$. The position vector of a material point in α^{th} microelement may be expressed as

(2.5) $$\underset{\sim}{X}^{(\alpha)} = \underset{\sim}{X} + \underset{\sim}{\Xi}^{(\alpha)}$$

where $\underset{\sim}{\Xi}^{(\alpha)}$ is the position of a point in the microelement relative to the center of mass of $\Delta V + \Delta S$, Fig. 2.2. Upon the deformation of the body, $\Delta V + \Delta S$ goes into $\Delta v + \Delta s$ with the microelement displaced with respect to the mass center. Because of the rearrangements and the relative deformations of the microelements, the center of mass P may now move to a position p and the material point Q to a new position q in the deformed body. The final position of the α^{th} particle will therefore be

(2.6) $$\underset{\sim}{x}^{(\alpha)} = \underset{\sim}{x} + \underset{\sim}{\xi}^{(\alpha)}$$

where $\dot{\underset{\sim}{x}}$ is the new position vector of the center of mass of Δv and $\underset{\sim}{\xi}^{(\alpha)}$ is the new relative position vector of the point originally located at $\underset{\sim}{X}^{(\alpha)}$. The motion of the center of mass P of ΔV is expressed as usual by (2.1) or simply

$$(2.7) \qquad \underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{X},t)$$

The relative position vector $\underset{\sim}{\xi}^{(\alpha)}$ however depends not only on $\underset{\sim}{X}$ and t but also on $\underset{\sim}{\Xi}^{(\alpha)}$, i.e.,

$$(2.8) \qquad \underset{\sim}{\xi}^{(\alpha)} = \underset{\sim}{\xi}^{(\alpha)}(\underset{\sim}{X},\underset{\sim}{\Xi}^{(\alpha)},t)$$

A microstructure theory must lean heavily on the assumption characterizing the nature of the functional character of $\underset{\sim}{\xi}^{(\alpha)}$. Eringen and Suhubi [1964a,b] and Eringen [1964] have constructed a general theory in which (2.8) is linear in $\underset{\sim}{\Xi}^{(\alpha)}$. The theory so constructed was later called by Eringen the *theory of micromorphic materials*. The basic assumption underlying this theory is the

Axiom of Affine Motion. *The material points in* $\Delta V + \Delta S$ *undergo a homogeneous deformation about the center of mass*, thus,

$$\underset{\sim}{\xi}^{(\alpha)} = \underset{\sim}{\chi}_1(\underset{\sim}{X},t)\Xi_1^{(\alpha)} + \underset{\sim}{\chi}_2(\underset{\sim}{X},t)\Xi_2^{(\alpha)} + \underset{\sim}{\chi}_3(\underset{\sim}{X},t)\Xi_3^{(\alpha)} \quad , \qquad \alpha = 1,2,\ldots N$$

or simply

$$(2.9) \qquad \underset{\sim}{\xi}^{(\alpha)} = \underset{\sim}{\chi}_K(\underset{\sim}{X},t)\Xi_K^{(\alpha)}$$

where summation over repeated indices is understood. This assumption is justifiable on the physical grounds that *when* $\Delta V + \Delta S$ *is small enough, its motion consists of a translation, a rotation about its center of mass, and a homogeneous deformation*. Note that in all classical theories of continuous media, this last assumption is missing (cf. Eringen [1962, Art. 10]).

Clearly, the theory can be extended to include quadratic and higher order terms in $\Xi^{(\alpha)}$. However, it quickly takes a very complex form, losing its simplicity and usefulness. In this article, in fact, we shall be concerned mostly with a much simpler case, namely, the <u>theory of micropolar elasticity</u>[1].

This latter theory admits only rigid microrotations of the microvolume elements about the center of mass of the volume element. In other words, the basic assumption (2.9) is further simplified by placing further restrictions on the three vector functions χ_K. As we shall see later, this will amount to reducing the number of the microdeformation functions χ_K from three to one. In fact, we shall also be dealing mainly with the linear theory. In classical continuum mechanics, the problem is centered around the determination of the spatial position x of all material points of the body at a given instant. This means that when we are through with all calculations, we will have the three functions $x_k(X,t)$ determined. In the theory of micromorphic materials, in addition the three vector functions $\chi_K(X,t)$ (equivalently nine scalar functions) must also be determined. The complicated nature of the problem and the necessity of additional physical concepts and laws are now apparent.

In coordinate form for the spatial position of a material point in a microelement, we have

(2.10) $$x_k^{(\alpha)} = x_k(X,t) + \chi_{kK}(X,t)\Xi_K^{(\alpha)}, \qquad k,K = 1,2,3$$

It is now clear that we have 3 + 9 = 12 functions, $x_k(X,t)$ and $\chi_{kK}(X,t)$, to determine the spatial position of the α^{th} material point $x_k^{(\alpha)}$.

Just as in (2.2), we introduce the <u>inverse micromotions</u> X_{Kk} such that

(2.11) $$\chi_{kK}X_{K\ell} = \delta_{k\ell}, \qquad \chi_{kK}X_{Lk} = \delta_{KL}$$

---

[1] The theory was developed in the original paper of Eringen and Suhubi [1964b, Sec. 6] as a special case of the general theory and was called the couple stress theory. Eringen [1965] later named it the micropolar theory.

Here and throughout this article, summation is understood to be over the repeated indices, e.g.,

$$x_{kK}X_{K\ell} \equiv x_{k1}X_{1\ell} + x_{k2}X_{2\ell} + x_{k3}X_{3\ell}$$

The symbols $\delta_{k\ell}$ and δ_{KL} are the Kronecker deltas which are 1 when the indices take the same numerical value and zero otherwise.

In component form (2.9) reads

$$\xi_k^{(\alpha)} = x_{kK}(\underset{\sim}{X},t)\Xi_K^{(\alpha)} \tag{2.12}$$

Upon multiplication of both sides by X_{Lk} and using (2.11), we also get

$$\Xi_K^{(\alpha)} = X_{Kk}(\underset{\sim}{x},t)\xi_k^{(\alpha)} \tag{2.13}$$

In vector form this reads

$$\underset{\sim}{\Xi}^{(\alpha)} = \underset{\sim}{X}_k(\underset{\sim}{x},t)\xi_k^{(\alpha)} \tag{2.14}$$

The rectangular components of $\underset{\sim}{X}_k$ are denoted by X_{Kk} and those of $\underset{\sim}{x}_K$ by x_{kK}, i.e.,

$$\underset{\sim}{x}_K = x_{kK}(\underset{\sim}{X},t)\underset{\sim}{i}_k \quad , \quad \underset{\sim}{X}_k = X_{Kk}(\underset{\sim}{x},t)\underset{\sim}{I}_K \tag{2.15}$$

where $\underset{\sim}{I}_K$ and $\underset{\sim}{i}_k$ are, respectively, the unit base vectors for the material coordinates X_K and the spatial coordinates x_k.

The motion and the inverse motion of a material point in a micro-element are therefore expressed by

$$x_k^{(\alpha)} = x_k(\underset{\sim}{X},t) + x_{kK}(\underset{\sim}{X},t)\Xi_K^{(\alpha)} \tag{2.16}$$

$$X_K^{(\alpha)} = X_K(\underset{\sim}{x},t) + X_{Kk}(\underset{\sim}{x},t)\xi_k^{(\alpha)} \tag{2.17}$$

In vectorial form these read

$$x^{(\alpha)} = x(X,t) + \chi_K(X,t)\Xi_K^{(\alpha)} \tag{2.18}$$

$$X^{(\alpha)} = X(x,t) + X_k(x,t)\xi_k^{(\alpha)} \tag{2.19}$$

It is clear that we may employ either representations (2.16) or (2.17), and that whenever either set of functions (x_k, χ_{kK}) or (X_K, X_{Kk}) are determined, the problem is completed since the other set is soluble from the one that is found.

3. STRAIN AND MICROSTRAIN TENSORS

The differential line element in the deformed body is calculated through (2.18)

$$\text{(3.1)} \qquad d\underset{\sim}{x}^{(\alpha)} = (\underset{\sim}{x},_K + \underset{\sim}{\chi}_{L,K}\Xi_L^{(\alpha)})dX_K + \underset{\sim}{\chi}_K d\Xi_K^{(\alpha)}$$

where an index followed by a comma denotes partial differentiation. This convention will be used throughout this article. Thus, for example,

$$\text{(3.2)} \qquad \underset{\sim}{x},_K \equiv \frac{\partial \underset{\sim}{x}}{\partial X_K} \quad , \quad \underset{\sim}{\chi}_{L,K} \equiv \frac{\partial \underset{\sim}{\chi}_L}{\partial X_K}$$

$$\text{(3.3)} \qquad \underset{\sim}{X},_k \equiv \frac{\partial \underset{\sim}{X}}{\partial x_k} \quad , \quad \underset{\sim}{\mathcal{X}}_{\ell,k} \quad \frac{\partial \underset{\sim}{\mathcal{X}}_\ell}{\partial x_k}$$

are used for brevity. Note that $(3.2)_1$ and $(3.3)_1$ are the classical deformation gradients, and $(3.2)_2$ and $(3.3)_2$ are the microdeformation gradients of the present theory.

The square of the arc length is now calculated by forming

$$(ds^{(\alpha)})^2 = d\underset{\sim}{x}^{(\alpha)} \cdot d\underset{\sim}{x}^{(\alpha)}$$

Upon using (3.1) and forming the scalar product, we find

$$\begin{aligned}\text{(3.4)} \quad (ds^{(\alpha)})^2 = {} & (C_{KL} + 2\,\Gamma_{KML}\Xi_M + \chi_{kM,K}\chi_{kN,L}\Xi_M\Xi_N)\; dX_K dX_L \\ & + 2(\Psi_{KL} + \chi_{kL}\chi_{kM,K}\Xi_M)\; dX_K d\Xi_L \\ & + \chi_{kK}\chi_{kL} d\Xi_K d\Xi_L\end{aligned}$$

where we also dropped the superscript α on Ξ_K and $d\Xi_K$ for brevity, since this is understood whenever we use the letter Ξ(and also ξ). In (3.4) we introduce the notations

$$C_{KL}(\underset{\sim}{X},t) \equiv x_{k,K}x_{k,L} \tag{3.5}$$

$$\Psi_{KL}(\underset{\sim}{X},t) \equiv x_{k,K}\chi_{kL} \tag{3.6}$$

$$\Gamma_{KLM}(\underset{\sim}{X},t) \equiv x_{k,K}\chi_{kL,M} \tag{3.7}$$

Of these, C_{KL} is the classical Green deformation tensor and Ψ_{KL} and Γ_{KLM} are new <u>microdeformation</u> <u>tensors</u> of the present theory.

We now introduce the displacement vector $\underset{\sim}{u}^{(\alpha)}$ as the vector that extends from $\underset{\sim}{X}^{(\alpha)}$ to $\underset{\sim}{x}^{(\alpha)}$, Fig. (3.1). Thus we write

$$\underset{\sim}{u}^{(\alpha)} = \underset{\sim}{x} - \underset{\sim}{X} + \underset{\sim}{\xi} - \underset{\sim}{\Xi} = \underset{\sim}{u} + \underset{\sim}{\xi} - \underset{\sim}{\Xi} \tag{3.8}$$

where

$$\underset{\sim}{u} \equiv \underset{\sim}{x} - \underset{\sim}{X} \tag{3.9}$$

is the classical displacement vector, the components of which in X_K and x_k are, respectively,

$$U_K \equiv \underset{\sim}{u} \cdot \underset{\sim}{I}_K = x_k\delta_{kK} - X_K \tag{3.10}$$

$$u_k \equiv \underset{\sim}{u} \cdot \underset{\sim}{i}_k = x_k - X_K\delta_{Kk} \tag{3.11}$$

where

$$\delta_{kK} \equiv \delta_{Kk} \equiv \underset{\sim}{i}_k \cdot \underset{\sim}{I}_K \tag{3.12}$$

Since the spatial and material frames are taken to be the same rectangular frame of reference, δ_{kK} is none other than the Kronecker delta which has the value one when the two indices take the same numerical value and zero otherwise. It is possible to write $x_k \delta_{kK} \equiv x_K$ and $X_K \delta_{Kk} \equiv X_k$. But we keep the convention of majuscule indices for the material frame and miniscule indices for the spatial frame of reference. This convention is espec ıv useful in the nonlinear theory.

From (3.10) and (3.11), by partial differentiation, we obtain

$$x_{k,K} = (\delta_{LK} + U_{L,K})\delta_{kL} \tag{3.13}$$

$$X_{K,k} = (\delta_{\ell k} - u_{\ell,k})\delta_{K\ell} \tag{3.14}$$

Similarly, we introduce the microdisplacement tensors $\Phi_{LK}(\underset{\sim}{X},t)$ and $\phi_{\ell k}(\underset{\sim}{x},t)$ by

$$\chi_{kK} = (\delta_{LK} + \Phi_{LK})\delta_{kL} \tag{3.15}$$

$$\mathcal{X}_{Kk} = (\delta_{\ell k} - \phi_{\ell k})\delta_{K\ell} \tag{3.16}$$

By use of (3.9) and (3.15) and (3.16), we see that (3.8) may also be expressed as

$$\underset{\sim}{u}^{(\alpha)} = \underset{\sim}{u} + \underset{\sim}{\xi} - \underset{\sim}{\Xi} = (U_K + \Phi_{KL}\Xi_L)\underset{\sim}{I}_K \tag{3.17}$$

$$\underset{\sim}{u}^{(\alpha)} = \underset{\sim}{u} + \underset{\sim}{\xi} - \underset{\sim}{\Xi} = (u_k + \phi_{k\ell}\xi_\ell)\underset{\sim}{i}_k \tag{3.18}$$

Upon substituting (3.13) and (3.15) into (3.5) to (3.7), we find that

$$C_{KL} = \delta_{KL} + U_{K,L} + U_{L,K} + U_{M,K}U_{M,L} \tag{3.19}$$

$$\Psi_{KL} = \delta_{KL} + \Phi_{KL} + U_{L,K} + U_{M,K}\Phi_{ML} \tag{3.20}$$

$$\Gamma_{KLM} = \Phi_{KL,M} + U_{N,K}\Phi_{NL,M} \tag{3.21}$$

So far all these expressions are exact. For a linear theory, one assumes that the product terms are negligible so that

$$C_{KL} \simeq \delta_{KL} + U_{K,L} + U_{L,K} \tag{3.22}$$

$$\Psi_{KL} \simeq \delta_{KL} + \Phi_{KL} + U_{L,K} \tag{3.23}$$

$$\Gamma_{KLM} \simeq \Phi_{KL,M} \tag{3.24}$$

In this case, the difference between the spatial and material representations disappears so that one may use u_k in place of U_K and $\phi_{k\ell}$ in place of Φ_{KL}, etc., a fact which is well known in the classical continuum theory (cf. Eringen [1962, Art. 14]). For the microdeformation, this may be seen as follows: If we use (3.15) and (3.16) in (2.11), we obtain

$$\Phi_{KL} = (\delta_{KM} + \Phi_{KM})\phi_{m\ell}\delta_{Mm}\delta_{\ell L} \tag{3.25}$$

Neglecting the product terms, we see that

$$\Phi_{KL} \simeq \phi_{m\ell}\delta_{Km}\delta_{\ell L} \tag{3.26}$$

which is the proof of our statement.

Since we will be dealing with the linear theory, we shall not distinguish between material and spatial representations except when it becomes necessary for clarity in the development.

The <u>material (or lagrangian) strain tensor</u> E_{KL} <u>and the material microstrain tensors</u> $\mathcal{E}_{KL}$ <u>and</u> Γ_{KLM} <u>are defined in the linear theory by</u>

$$E_{KL} \equiv \frac{1}{2}(C_{KL} - \delta_{KL}) = \frac{1}{2}(U_{K,L} + U_{L,K}) \tag{3.27}$$

$$\mathcal{E}_{KL} \equiv \Psi_{KL} - \delta_{KL} = \Phi_{KL} + U_{L,K} \tag{3.28}$$

$$\Gamma_{KLM} = \Phi_{KL,M} \tag{3.29}$$

In the light of what has been shown above, we may also introduce the spatial (or eulerian) strain tensor $e_{k\ell}$ and spatial microstrain tensors, $\varepsilon_{k\ell}$ and $\gamma_{k\ell m}$ in a similar fashion:

(3.30) $$e_{k\ell} \equiv \frac{1}{2}(u_{k,\ell} + u_{\ell,k})$$

(3.31) $$\varepsilon_{k\ell} \equiv \phi_{k\ell} + u_{\ell,k}$$

(3.32) $$\gamma_{k\ell m} = - \phi_{k\ell,m}$$

Clearly, when these tensors are known, changes in arc length and angles during any deformation of the body can be calculated.

For the linear theory, the difference between the squares of arc length in the deformed and undeformed body follows from (3.4) and using (3.15) and (3.27) to (3.29)

(3.33) $$\begin{aligned}(ds^{(\alpha)})^2 - (dS^{(\alpha)})^2 &\simeq 2(E_{KL} + \Gamma_{KML}\Xi_M)dX_K dX_L \\ &+ 2(\mathcal{E}_{KL} + \Gamma_{LMK}\Xi_M)dX_K d\Xi_L \\ &+ (\mathcal{E}_{KL} + \mathcal{E}_{LK} - 2E_{KL})d\Xi_K d\Xi_L\end{aligned}$$

In classical continuum mechanics, only the first term on the right involving E_{KL} survives.

From (3.33) it is clear that when E_{KL}, $\mathcal{E}_{KL}$ and Γ_{KLM} vanish, there will be no change in the arc length after a deformation. In such a situation, the body is said to undergo a rigid motion.

4. MICROPOLAR STRAINS AND ROTATIONS

We now consider a special class of materials in which the state of the microdeformation can be described by a local rigid motion of the microelements. A large class of materials exist in which the micro-material elements are dumbbell type molecules. Materials consisting of rigid fibers or elongated grains fall into this category. For example, wood, certain rocks, and minerals contain elongated molecular elements. Among fluids, blood possesses dumbbell-like molecules. For such media, the micromorphic material theory becomes much simpler. Mathematically, this specialization is obtained by setting

$$\Phi_{KL} = - \Phi_{LK} \tag{4.1}$$

or in the spatial notation $\phi_{k\ell} = - \phi_{\ell k}$. In three-dimensional space, every skew-symmetric second-order tensor Φ_{KL} can be expressed by an axial vector Φ_K defined by

$$\Phi_K = \frac{1}{2} \varepsilon_{KLM} \Phi_{ML} \tag{4.2}$$

where ε_{KLM} is the alternating tensor defined as

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = -\varepsilon_{213} = -\varepsilon_{132} = -\varepsilon_{321} = 1,$$

$$\varepsilon_{KLM} = 0 \text{ otherwise}$$

Expression (4.2) is a compact expression of

$$\Phi_1 = \Phi_{32}, \ \Phi_2 = \Phi_{13}, \ \Phi_3 = \Phi_{21}$$

The solution of (4.2) for Φ_{KL} is given by

$$\Phi_{KL} = -\varepsilon_{KLM} \Phi_M \tag{4.3}$$

Substituting this into (3.15), we see that

(4.4) $$\chi_{kK} = \delta_{kK} - \varepsilon_{kKM}\Phi_M$$

In the classical theory we have the rotation tensor

(4.5) $$R_{KL} = -R_{LK} \equiv \frac{1}{2}\,(U_{K,L} - U_{L,K})$$

The axial vector R_k corresponding to this is given by

(4.6)
$$R_K = \frac{1}{2}\,\varepsilon_{KLM}R_{ML} = \frac{1}{2}\,\varepsilon_{KLM}U_{M,L},$$
$$R_{KL} = -\,\varepsilon_{KLM}R_M$$

Using (3.27) and $(4.6)_2$, we find that

(4.7) $$U_{K,L} = E_{KL} + R_{KL} = E_{KL} - \varepsilon_{KLM}R_M$$

When this and (4.2) are substituted into (3.28) and $(3.29)_1$, we get

(4.8) $$\mathcal{E}_{KL} = E_{KL} + \varepsilon_{KLM}(R_M - \Phi_M)$$

(4.9) $$\Gamma_{KLM} = -\varepsilon_{KLN}\Phi_{N,M}$$

When $R_M = \Phi_M$, wee see that $\mathcal{E}_{KL} = E_{KL}$ and $\Gamma_{KLM} = R_{KL,M}$ and the microstrains are no longer independent of the classical strains and rotations. Thus the micropolar theory assumes that <u>the classical rotation R_K is different from the microrotation</u>. In the micropolar theory, we have therefore six functions to determine, namely $U_K(\underset{\sim}{X},t)$ and $\Phi_K(\underset{\sim}{X},t)$. Once this is done, length and angle changes can be fully calculated.

For the micropolar theory, the spatial position of the α^{th} point $\underset{\sim}{x}^{(\alpha)}$ is obtained through (3.8), (3.18), and (4.3), i.e., (4.10) $$\underset{\sim}{x}^{(\alpha)} = \underset{\sim}{X} + \underset{\sim}{\Xi} + \underset{\sim}{u} - \underset{\sim}{\Xi} \times \underset{\sim}{\Phi}$$

From this it is apparent that $\underset{\sim}{\Phi}$ represents an angular rotation of a microelement about the center of mass of the deformed macrovolume element ($\underset{\sim}{\xi}$ is the moment arm from this centroid), Fig. 4.1. Accordingly, we also have

(4.11) $$\underset{\sim}{\xi} = \underset{\sim}{\Xi} - \underset{\sim}{\Xi} \times \underset{\sim}{\Phi}$$

which shows that aside from a rigid body translation the relative position of $\underset{\sim}{\Xi}$ of a material point after deformation is obtained by translating $\underset{\sim}{\Xi}$ parallel to itself to the center of mass $\underset{\sim}{x}$ of the deformed macrovolume element, and then rotating it an amount $\underset{\sim}{\Xi} \times \underset{\sim}{\Phi}$. Since we are dealing with linear theory, we also have

(4.12) $$\underset{\sim}{\Xi} = \underset{\sim}{\xi} + \underset{\sim}{\xi} \times \underset{\sim}{\phi}$$

where $\underset{\sim}{\phi} \simeq \underset{\sim}{\Phi}$ is the spatial microrotation. In fact, we have the complete dual to equations (4.1) to (4.10) which we record here for future convenience.

(4.13) $$\phi_k = \frac{1}{2}\,\varepsilon_{k\ell m}\phi_{m\ell} \quad , \quad \phi_{k\ell} = -\varepsilon_{k\ell m}\phi_m$$

(4.14) $$\chi_{K\ell} = \delta_{K\ell} - \varepsilon_{K\ell m}\phi_m$$

(4.15) $$r_k = \frac{1}{2}\,\varepsilon_{k\ell m}r_{m\ell} \, , \quad r_{k\ell} = -\varepsilon_{k\ell m}r_m = \frac{1}{2}\,\varepsilon_{k\ell m}u_{m,\ell}$$

(4.16) $$u_{k,\ell} = e_{k\ell} - e_{k\ell m}r_m$$

(4.17) $$\varepsilon_{k\ell} = e_{k\ell} + \varepsilon_{k\ell m}(r_m - \phi_m)$$

(4.18) $$\gamma_{k\ell m} = \varepsilon_{k\ell n}\phi_{n,m}$$

and

$$x^{(\alpha)} = X + \Xi + u - \xi \times \phi \tag{4.19}$$

Now consider the deformation of an infinitesimal vector $dX^{(\alpha)} = dX + d\Xi$ at $X + \Xi$. Upon deformation, this vector becomes

$$dx^{(\alpha)} = dx + d\xi = dX + d\Xi + u_{,K}dX_K - d\Xi \times \phi - \Xi \times \phi_{,K}dX_K \tag{4.20}$$

By use of (4.7) and (4.6) we may write

$$\begin{aligned} u_{,K}dX_K &= U_{L,K}dX_K I_L = E_{LK}dX_K I_L + R_{LK}dX_K I_L \\ &= E_{KL}dX_K I_L - dX \times R \end{aligned} \tag{4.21}$$

Similarly using (4.9)

$$\Xi \times \phi_{,N}dX_N = \varepsilon_{KLM}\Xi_L\phi_{M,N}dX_N I_K = -\Gamma_{KLN}\Xi_L dX_N I_K$$

For convenience we now introduce the notation

$$\Gamma_{KM} \equiv \Gamma_{KLM}\Xi_L \tag{4.22}$$

so that

$$\Xi \times \phi_{,N}dX_N = -\Gamma_{(KM)}dX_M I_K - \Gamma_{[KM]}dX_M I_K \tag{4.23}$$

where indices in parenthesis (and brackets) indicate the symmetric (and antisymmetric) parts. Carrying (4.21) and (4.23) into (4.20) we rearrange it into

$$\begin{aligned} dx^{(\alpha)} &= dX + d\Xi - (dX \times R + d\Xi \times \phi + dX \times \Gamma) \\ &\quad + (E_{KL} + \Gamma_{(KL)})dX_K I_L \end{aligned} \tag{4.24}$$

where we also defined a new microrotation vector Γ by

$$\Gamma_K \equiv \frac{1}{2}\varepsilon_{KLM}\Gamma_{ML} \quad , \quad \Gamma_{[MK]} = -\varepsilon_{KLM}\Gamma_M \tag{4.25}$$

We name this vector minirotation for distinction from the microrotation $\underset{\sim}{\Phi}$. If we carry (4.9) into $(4.25)_1$, we also find

$$\Gamma_K = \frac{1}{2}(-\Phi_{L,L}\Xi_K + \Phi_{K,L}\Xi_L) \tag{4.26}$$

Equation (4.24) reveals that the deformation of the vector $d\underset{\sim}{X}^{(\alpha)} \equiv d\underset{\sim}{X} + d\underset{\sim}{\Xi}$ may be achieved by the following three operations:

(a) A rigid translation of $d\underset{\sim}{X} + d\underset{\sim}{\Xi}$ from the material centroid $\underset{\sim}{X}$ to the spatial centroid $\underset{\sim}{x}$.

(b) Rigid rotations of $d\underset{\sim}{X}$ and $d\underset{\sim}{\Xi}$ by the amounts $d\underset{\sim}{X} \times (\underset{\sim}{R} + \underset{\sim}{\Gamma})$ and $d\underset{\sim}{\Xi} \times \underset{\sim}{\Phi}$, respectively.

(c) Finally, a stretch represented by the strains E_{KL} and $\Gamma_{(KL)}$ in (4.24).

The following special cases help to visualize these deformations:

(i) When $\underset{\sim}{\Xi} \equiv \underset{\sim}{0}$ we have

$$d\underset{\sim}{x}^{(\alpha)} = d\underset{\sim}{x} = d\underset{\sim}{X} - d\underset{\sim}{X} \times \underset{\sim}{R} + E_{KL}dX_K\underset{\sim}{I}_L \tag{4.27}$$

This, of course, is a well-known theorem in classical continuum mechanics attributed to Helmholtz (cf. Eringen [1962, Art. 10]).

(ii) When $\underset{\sim}{\Xi}$ = constant vector

$$d\underset{\sim}{x}^{(\alpha)} = d\underset{\sim}{X} - d\underset{\sim}{X} \times (\underset{\sim}{R} + \underset{\sim}{\Gamma}) + (E_{KL} + \Gamma_{(KL)})dX_K\underset{\sim}{I}_L \tag{4.28}$$

Here, of course, we have no rotation of $d\underset{\sim}{\Xi} \equiv 0$.

(iii) $\underset{\sim}{\Phi}$ = constant microrotation. In this case we have $\Gamma_{(KL)} = \Gamma_{[KL]} = \Gamma_K = 0$ and we obtain

(4.29) $$d\underset{\sim}{x}^{(\alpha)} = d\underset{\sim}{X} + d\underset{\sim}{\Xi} - d\underset{\sim}{X} \times \underset{\sim}{R} + d\underset{\sim}{\Xi} \times \underset{\sim}{\Phi} + E_{KL}dX_K\underset{\sim}{I}_L$$

In this case, the rotation consists only of a macrorotation of $d\underset{\sim}{X}$ and a microrotation of $d\underset{\sim}{\Xi}$. We can also write it as

(4.30) $$d\underset{\sim}{\xi} = d\underset{\sim}{x}^{(\alpha)} - d\underset{\sim}{x} = d\underset{\sim}{\Xi} - d\underset{\sim}{X} \times \underset{\sim}{\Gamma} - d\underset{\sim}{\Xi} \times \underset{\sim}{R} + \Gamma_{(KL)}dX_K\underset{\sim}{I}_L$$

In this form we see the difference between the deformation of $d\underset{\sim}{X}^{(\alpha)}$ and that of $d\underset{\sim}{X}$ of which the latter is known to us from the classical theory. This difference, therefore, is the result of the composition of a minirotation of $d\underset{\sim}{X}$, a macrorotation of $d\underset{\sim}{\Xi}$, and the ministraining of $d\underset{\sim}{X}$ characterized by $\Gamma_{(KL)}$. The terminology of minirotation is being used for $\underset{\sim}{\Gamma}$ and ministraining for $\Gamma_{(KL)}$. Of course, $\underset{\sim}{R}$ is the classical rotation for which we use the terminology macrorotation.

For the spatial representation the dual of (4.10) is

(4.31) $$\underset{\sim}{X}^{(\alpha)} = \underset{\sim}{x} + \underset{\sim}{\xi} - \underset{\sim}{u} + \underset{\sim}{\xi} \times \underset{\sim}{\phi}$$

From this, in the same way as in the case of (4.24), we obtain

(4.32) $$d\underset{\sim}{X}^{(\alpha)} = d\underset{\sim}{x} + d\underset{\sim}{\xi} + (d\underset{\sim}{x} \times \underset{\sim}{r}) + d\underset{\sim}{\xi} \times \underset{\sim}{\phi} + d\underset{\sim}{x} \times \underset{\sim}{\gamma}) - (e_{k\ell} - \gamma_{(k\ell)})dx_k\underset{\sim}{i}_\ell$$

where $\underset{\sim}{r}$ is the spatial macrorotation vector defined by (4.15) and $e_{k\ell}$ is the spatial macrostrain tensor. The spatial minirotation vector γ is given by

(4.33) $$\gamma_k = \frac{1}{2}\varepsilon_{k\ell m}\gamma_{m\ell} = \frac{1}{2}(\phi_{\ell,\ell}\xi_k - \phi_{k,\ell}\xi_\ell)$$

and

(4.34) $$\gamma_{km} \equiv \gamma_{k\ell m}\xi_\ell = \varepsilon_{k\ell n}\phi_{n,m}\xi_\ell$$

Finally, below we record the component form of various strains and rotations in spatial rectangular coordinates x, y, z. For the components of the displacement vector $\underset{\sim}{u}$ and micropolar rotation vector $\underset{\sim}{\phi}$ we write, respectively, u, v, w and ϕ_x, ϕ_y, ϕ_z.

Macrostrain tensor:

$$
\begin{aligned}
& e_{xx} = \frac{\partial u}{\partial x} \quad , \quad e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\
(4.35) \qquad & e_{yy} = \frac{\partial v}{\partial y} \quad , \quad e_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\right) \\
& e_{zz} = \frac{\partial w}{\partial z} \quad , \quad e_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)
\end{aligned}
$$

Micropolar strain tensor:

$$
\begin{aligned}
& \varepsilon_{xx} = \frac{\partial u}{\partial x} \quad , \quad \varepsilon_{yy} = \frac{\partial v}{\partial y} \quad , \quad \varepsilon_{zz} = \frac{\partial w}{\partial z} \\
& \varepsilon_{xy} = \frac{\partial v}{\partial x} - \phi_z \quad , \quad \varepsilon_{yx} = \frac{\partial u}{\partial y} + \phi_z \\
(4.36) \qquad & \varepsilon_{yz} = \frac{\partial w}{\partial y} - \phi_x \quad , \quad \varepsilon_{zy} = \frac{\partial v}{\partial z} + \phi_x \\
& \varepsilon_{zx} = \frac{\partial u}{\partial z} - \phi_y \quad , \quad \varepsilon_{xz} = \frac{\partial w}{\partial x} + \phi_y
\end{aligned}
$$

Micropolar strain tensor of third order:

$$\gamma_{yzx} = -\gamma_{zyx} = \frac{\partial\phi_x}{\partial x}, \quad \gamma_{yzy} = -\gamma_{zyy} = \frac{\partial\phi_x}{\partial y}, \quad \gamma_{yzz} = -\gamma_{zyz} = \frac{\partial\phi_x}{\partial z}$$

$$(4.37) \quad \gamma_{zxx} = -\gamma_{xzx} = \frac{\partial\phi_y}{\partial x}, \quad \gamma_{zxy} = -\gamma_{xzy} = \frac{\partial\phi_y}{\partial y}. \quad \gamma_{zxz} = -\gamma_{xzz} = \frac{\partial\phi_y}{\partial z}$$

$$\gamma_{xyz} = -\gamma_{yxx} = \frac{\partial\phi_z}{\partial x}, \quad \gamma_{xyy} = -\gamma_{yxy} = \frac{\partial\phi_z}{\partial y}, \quad \gamma_{xyz} = -\gamma_{yxz} = \frac{\partial\phi_z}{\partial z}$$

all other $\gamma_{k\ell m} = 0$.

Macrorotation vector:

$$r_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$(4.38) \quad r_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$r_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Microrotation vector:

$$(4.39) \quad \underset{\sim}{\phi} = \phi_x \underset{\sim}{i} + \phi_y \underset{\sim}{j} + \phi_z \underset{\sim}{k}$$

Minirotation vector:

$$\gamma_x = \frac{1}{2}\left[\left(\frac{\partial\phi_y}{\partial y} + \frac{\partial\phi_z}{\partial z}\right)\xi_x - \frac{\partial\phi_x}{\partial y}\xi_y - \frac{\partial\phi_x}{\partial z}\xi_z\right]$$

$$(4.40) \quad \gamma_y = \frac{1}{2}\left[\left(\frac{\partial\phi_x}{\partial x} + \frac{\partial\phi_z}{\partial z}\right)\xi_y - \frac{\partial\phi_y}{\partial x}\xi_x - \frac{\partial\phi_y}{\partial z}\xi_z\right]$$

$$\gamma_z = \frac{1}{2}\left[\left(\frac{\partial\phi_x}{\partial x} + \frac{\partial\phi_y}{\partial y}\right)\xi_z - \frac{\partial\phi_z}{\partial x}\xi_x - \frac{\partial\phi_z}{\partial y}\xi_y\right]$$

5. GEOMETRICAL MEANING OF MICROPOLAR STRAINS AND ROTATIONS

The geometrical significance of various strain and rotation measures can be understood quickly if we consider the change $d\underset{\sim}{x}^{(\alpha)}$ in $\underset{\sim}{x}^{(\alpha)}$ as the vector sum of three changes, namely,

$$d\underset{\sim}{x}^{(\alpha)} = d\underset{\sim}{x} + d\underset{\sim}{y} + d\underset{\sim}{x} \tag{5.1}$$

where

$$d\underset{\sim}{x} = d\underset{\sim}{X} - d\underset{\sim}{X} \times \underset{\sim}{R} + E_{KL}dX_K\underset{\sim}{I}_L \tag{5.2}$$

$$d\underset{\sim}{y} = -d\underset{\sim}{X} \times \underset{\sim}{\Gamma} + \Gamma_{(KL)}dX_K\underset{\sim}{I}_L \tag{5.3}$$

$$d\underset{\sim}{z} = d\underset{\sim}{\Xi} - d\underset{\sim}{\Xi} \times \underset{\sim}{\phi} \tag{5.4}$$

Here $d\underset{\sim}{x}$ is that known to us from classical continuum theory (cf. Eringen [1962, Art. 6]). Accordingly, on the right-hand side of (5.2) the first term $d\underset{\sim}{X}$ represents translation of $d\underset{\sim}{x}$ from $\underset{\sim}{X}$ to $\underset{\sim}{x}$; the second is a rotation; and the last term represents the straining of the body. More specifically, consider the vector $d\underset{\sim}{X}$ at the point $\underset{\sim}{X}$ of the undeformed volume element dV. This vector after deformation becomes $d\underset{\sim}{x}$.

Writing (5.2) in another form

$$d\underset{\sim}{x} = \underset{\sim}{C}_K dX_K \tag{5.5}$$

where

$$\underset{\sim}{C}_K \equiv \frac{\partial \underset{\sim}{x}}{\partial X_K} = \underset{\sim}{I}_K + U_{M,K}\underset{\sim}{I}_M \tag{5.6}$$

we see that a parallelepiped with side vectors $\underset{\sim}{I}_1 dX_1$, $\underset{\sim}{I}_2 dX_2$, and $\underset{\sim}{I}_3 dX_3$ after deformation becomes a rectilinear parallelepiped with side vectors

$\underset{\sim}{C}_1 dX_1$, $\underset{\sim}{C}_2 dX_2$ and $\underset{\sim}{C}_3 dX_3$, Fig. 5.1. The <u>stretch</u> $\Lambda_{(\underset{\sim}{N})}$ and <u>extension</u> $E_{(\underset{\sim}{N})}$ are defined by

(5.7) $$E_{(\underset{\sim}{N})} \equiv \Lambda_{(\underset{\sim}{N})} - 1 \equiv \frac{|d\underset{\sim}{x}| - |d\underset{\sim}{X}|}{|d\underset{\sim}{X}|}$$

Now calculate the extension of one of the sides of the undeformed parallelepiped at $\underset{\sim}{X}$, e.g., $\underset{\sim}{I}_1 dX_1$:

$$|d\underset{\sim}{x}|^2 = \underset{\sim}{C}_1 dX_1 \cdot \underset{\sim}{C}_1 dX_1 = C_{11}(dX_1)^2$$

Hence

$$E_{(1)} = \Lambda_{(1)} - 1 = \sqrt{C_{11}} - 1$$

But we have

$$C_{11} = 1 + 2E_{11}$$

Thus

(5.8) $$E_{(1)} = \Lambda_{(1)} - 1 = \sqrt{1 + 2E_{11}} - 1$$

From this it follows that

$$2E_{11} = (1 + E_{(1)})^2 - 1$$

For small extensions $E_{(1)} << 1$, and this approximates to

(5.9) $$E_{11} \approx E_{(1)}$$

which provides a meaning for the <u>normal components</u> E_{11}, E_{22}, and E_{33} of the infinitesimal strain tensor. For the shear strains E_{12}, E_{23}, E_{31} we find a geometrical meaning by calculating the change of angle between two side vectors such as $\underset{\sim}{I}_1 dX_1$ and $\underset{\sim}{I}_2 dX_2$. The angle $\theta_{(12)}$ between these two vectors after deformation is calculated by

$$\cos \theta_{(12)} = \frac{\underset{\sim}{C}_1 dX_1 \cdot \underset{\sim}{C}_2 dX_2}{|\underset{\sim}{C}_1| dX_1 |\underset{\sim}{C}_2| dX_2} = \frac{C_{12}}{\sqrt{C_{11} C_{22}}}$$

$$= \frac{2E_{12}}{\sqrt{1 + 2E_{11}}\ \sqrt{1 + 2E_{22}}}$$

The change of angle $\Gamma_{(12)} = \frac{\pi}{2} - \theta_{(12)}$ between the original and the final angle follows from this

$$\sin \Gamma_{(12)} = \frac{2E_{12}}{\sqrt{1 + 2E_{11}}\ \sqrt{1 + 2E_{22}}} \tag{5.10}$$

For small strains $E_{11} << 1$, $E_{22} << 1$ and $\sin \Gamma_{(12)} \simeq \Gamma_{(12)}$ so that we have the approximation

$$\Gamma_{(12)} \simeq 2E_{12} \tag{5.11}$$

which provides the geometrical meaning for the shear strain E_{12}. Similar results are, of course, valid for E_{23} and E_{31}.

For the rotation vector $\underset{\sim}{R}$ we have the following geometrical interpretation: Let $\underset{\sim}{N}_3$ be a unit vector in the X_1, X_2-plane at $\underset{\sim}{X}$. After deformation, $\underset{\sim}{N}_3$ becomes a vector $\underset{\sim}{n}_3$ at $\underset{\sim}{x}$. Bring $\underset{\sim}{n}_3$ to $\underset{\sim}{X}$ and obtain its projection $\underset{\sim}{n}_3^*$ on X_1, X_2-plane, Fig. 5.2. Let the angle between $\underset{\sim}{N}_3$ and $\underset{\sim}{n}_3^*$ be denoted by θ_3. We can show that (cf. Eringen [1962, Art. 10]) the average $< \tan \theta_3 >$ over-all angles Φ that $\underset{\sim}{N}_3$ can make with the X_1-axis is related to R_3 by

$$< \tan \theta_3 > = \frac{R_3}{\sqrt{(1 + E_{11})(1 + E_{22}) - E_{12}^2}} \tag{5.12}$$

where E_{11}, E_{22}, and E_{12} are the infinitesimal strain components and R_3 is the rotation component in the X_3-direction, i.e.,

$$R_3 = -R_{12} = \frac{1}{2}\left(\frac{\partial U_2}{\partial X_1} - \frac{\partial U_1}{\partial X_2}\right)$$

For small strains (5.12) can be approximated to give

$$\langle \theta_3 \rangle \simeq R_3 \tag{5.13}$$

which provides a geometrical meaning for R_3. Similar interpretations are, of course, valid for R_1 and R_2. Accordingly, for small deformations and rotations, $\underset{\sim}{R}$ represents the average rotation angle of the diagonal vector $d\underset{\sim}{X}$ of the undeformed parallelepiped.

The above discussion concerning the deformed parallelpiped with diagonal $d\underset{\sim}{x}$ at $\underset{\sim}{x}$ may be used to provide a geometrical meaning for the deformed parallelepiped with diagonal $d\underset{\sim}{y}$ at $\underset{\sim}{x}$. To this end, from (5.3) we observe that $\underset{\sim}{\Gamma}$ takes the place of the rotation $\underset{\sim}{R}$ and $\Gamma_{(KL)}$ that of E_{KL}. Accordingly, the undeformed parallelepiped with diagonal vector at $\underset{\sim}{X} + \underset{\sim}{\Xi}$ with $\underset{\sim}{\Xi}$ fixed, undergoes an additional rotation $\underset{\sim}{\Gamma}$ and length and angle changes represented by the strains $\Gamma_{(KL)}$ at the spatial point $\underset{\sim}{x} + \underset{\sim}{\xi}$ that the point $\underset{\sim}{X} + \underset{\sim}{\Xi}$ occupies after deformation. This deformation takes place with $\underset{\sim}{\Xi}$ being displaced parallel to itself. This deformation emanates primarily from the parallelepiped with diagonal $d\underset{\sim}{X}$ being constructed at the point $\underset{\sim}{X} + \underset{\sim}{\Xi}$ with $\underset{\sim}{\Xi}$ fixed instead of being at $\underset{\sim}{X}$, as in the first case.

Finally, a parallelepiped with diagonal $d\underset{\sim}{\Xi}$ constructed at $\underset{\sim}{X} + \underset{\sim}{\Xi}$ with $\underset{\sim}{X}$ fixed is rigidly displaced to a spatial point $\underset{\sim}{x} + \underset{\sim}{\xi}$ and rotated with the microrotation $\underset{\sim}{\phi}$ at that point. Thus the final rectilinear parallelepiped with diagonal $dx^{(\alpha)}$ at $\underset{\sim}{x} + \underset{\sim}{\xi}$ is constructed as a result of these translations, rotations, and strains.

The picture of the deformation in these latter cases is clarified further if we write (cf. (3.1))

$$\text{(5.14)} \qquad d\underset{\sim}{x}^{(\alpha)} = (\underset{\sim}{C}_K + \underset{\sim}{C}_K)dX_K + \underset{\sim}{\chi}_K d\Xi_K$$

where

$$\text{(5.15)} \qquad \underset{\sim}{C}_K \equiv \underset{\sim}{\chi}_{L,K}\Xi_L$$

and $\underset{\sim}{\chi}_K$ is related to ϕ_{LK} by (3.15).

In the form (5.14), the deformation of a parallelepiped with diagonal $d\underset{\sim}{X}$ at $\underset{\sim}{X} + \underset{\sim}{\Xi}$ and another one with $d\underset{\sim}{\Xi}$ are shown on Figure 5.3. According to this picture, $d\underset{\sim}{X}$ at $\underset{\sim}{X} + \underset{\sim}{\Xi}$ with fixed $\underset{\sim}{\Xi}$ becomes $(\underset{\sim}{C}_K + \underset{\sim}{C}_K)dX_K$, Fig. 5.4, and the one with diagonal $d\underset{\sim}{\Xi}$ at $\underset{\sim}{X} + \underset{\sim}{\Xi}$ with $\underset{\sim}{X}$ fixed becomes $\underset{\sim}{\chi}_K dX_K$, Fig. 5.5. The resulting deformation of $d\underset{\sim}{X} + d\underset{\sim}{\Xi}$ at $\underset{\sim}{X} + \underset{\sim}{\Xi}$ is the vector sum of these two deformations, Fig. 5.3.

6. INVARIANTS OF STRAIN TENSORS

The state of local deformation at a point $\underset{\sim}{X} + \underset{\sim}{\Xi}$ of a micromorphic material is fully determined when the three material strain tensors

$$E_{KL}\ ,\ \mathcal{E}_{KL}\ ,\ \Gamma_{KLM} \tag{6.1}$$

are given. With knowledge of these tensors, we can calculate length (cf. (3.30)) and angle changes (Art. 5) and construct the spatial locations of various material points in the body. Instead of the list (6.1), we may of course employ the spatial tensors $e_{k\ell}$, $\varepsilon_{k\ell}$ and $\gamma_{k\ell m}$. An important question in continuum mechanics is: If the material (spatial) coordinates are rigidly rotated at $\underset{\sim}{X}$ (at $\underset{\sim}{x}$), are there some functions of the material (spatial) strain measures which remain unchanged? The answer to this question is provided by the <u>theory of invariants</u>. In fact, the theory of invariants is concerned with a more difficult question, namely: To determine the complete set of invariants (called <u>integrity basis</u>) of a given set of vectors and tensors which are unchanged under an arbitrary group of transformations of coordinates. The <u>minimal basis</u> is a subset of these invariants which can be employed to express all other invariants of a given set. The answer to this question is important on two accounts:

(i) Constitutive equations must be form-invariant under rigid motions of the spatial frame of reference. This is known as the <u>principle of objectivity</u>. Investigation of this restriction on the constitutive equations often requires knowledge of the invariants of constitutive variables such as strain measures.

(ii) The Material symmetry places restrictions on the form of the constitutive equations when the material frame of reference is transformed according to some group of transformations. For example, when the material has a plane of symmetry, the constitutive equations should not change their forms when a reflection of axes is performed according to the plane of symmetry. Similarly, for isotropic materials, the constitutive equations remain form-invariant under the full group of orthogonal transformations of the material frame of reference. For anisotropic materials with some symmetry axes, the group of transformations is less restrictive.

The invariants of a symmetric second-order tensor (such as E_{KL}) in three dimensions are:

$$(6.2) \qquad I_E \equiv E_{KK} \;,\quad II_E \equiv \frac{1}{2}(E_{KK}E_{LL} - E_{KL}E_{LK}) \;,\quad III_E \equiv \det E_{KL}$$

Instead of this set, one may also employ

$$(6.3) \qquad \begin{aligned} \operatorname{tr} \underset{\sim}{E} &\equiv E_{KL} \\ \operatorname{tr} \underset{\sim}{E}^2 &\equiv E_{KL}E_{LK} \\ \operatorname{tr} \underset{\sim}{E}^3 &\equiv E_{KL}E_{LM}E_{MK} \end{aligned}$$

The above two sets are related to each other. In fact

$$(6.4) \qquad \begin{aligned} &\operatorname{tr} \underset{\sim}{E} = I_E \;,\qquad \operatorname{tr} \underset{\sim}{E}^2 = I_E^2 - 2II_E \\ &\operatorname{tr} \underset{\sim}{E}^3 = I_E^3 - 3I_E II_E + 3III_E \end{aligned}$$

The determination of the minimal integrity basis of (6.1) is much more complicated. In fact, to our knowledge the basic invariants of third and higher-order tensors have not been studied to date. Fortunately, in the present theory Γ_{KLM} always occur in the form $\Gamma_{KLM}\Xi_L \equiv \Gamma_{KM}$ so that we may instead search for the invariants of

$$(6.5) \qquad E_{KL} \;,\; \mathcal{E}_{KL} \;,\; \Gamma_{KL}$$

For the micropolar theory, the situation is simplified further since $\mathcal{E}_{(KL)} = E_{KL}$. The invariants of two symmetric and two antisymmetric second-order tensors

$$(6.6) \qquad \mathcal{E}_{(KL)} = E_{KL} \quad , \quad \mathcal{E}_{[KL]} \quad , \quad \Gamma_{(KL)} \quad , \quad \Gamma_{[KL]}$$

would be sufficient for this purpose. The integrity basis for the proper orthogonal group for such a set has been studied by various authors. Below we give a table for the construction of these invariants. For the sake of simplicity, we introduce the symbols $\underset{\sim}{a}$, $\underset{\sim}{b}$ for the symmetric tensors and $\underset{\sim}{u}$, $\underset{\sim}{v}$ for the antisymmetric tensors. In Table 6.1 we give the invariants of these tensors in ascending order of the integrity basis of various subsets of $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{u}$ and $\underset{\sim}{v}$. The integrity basis of the quantities listed in each entry of the first column includes all entries on this row and the integrity basis of all subsets of these quantities. Thus, for example, the integrity basis for $\underset{\sim}{a}$, $\underset{\sim}{b}$ in the second row includes that of $\underset{\sim}{a}$ and $\underset{\sim}{b}$, namely, tr $\underset{\sim}{a}$, tr $\underset{\sim}{a}^2$, tr $\underset{\sim}{a}^3$, and tr $\underset{\sim}{b}$, tr $\underset{\sim}{b}^2$, tr $\underset{\sim}{b}^3$. Also an asterisk (*) placed on the products indicates that we include in this lisr all other products obtained from this by cyclic permutations of the symmetric matrices. A dagger (†) indicates the inclusion of all quantities obtained by cyclic permutation of the skew-symmetric matrices. Thus, for example, ab* means the inclusion of the set

$$\underset{\sim}{a}\underset{\sim}{b} \ , \ \underset{\sim}{b}\underset{\sim}{a}$$

Similarly, $\underset{\sim}{u}^2\underset{\sim}{v}\underset{\sim}{a}^\dagger$ means the inclusion of the set

$$\underset{\sim}{u}^2\underset{\sim}{v}\underset{\sim}{a}^\dagger \ , \ \underset{\sim}{v}\underset{\sim}{u}^2\underset{\sim}{a}^\dagger$$

For other details and more extensive studies on invariant theory see Spencer [1966]. Since we will be interested, generally, in a linear theory, many entries in this list will not be needed in the construction of the constitutive equations.

TABLE 6.1

Matrix products whose traces form the integrity basis in the proper orthogonal group

Matrices	Matrix Products
a	a; a^2; a^3
a, b	ab; ab^{2*}; a^2b^2
u	u^2
u, a	u^2a; u^2a^2; u^2aua^2
u, a, b	uab; ua^2b^*; ua^2b^{2*}; ua^2ba^*; $ua^2b^2a^*$; u^2ab; $u^2a^2b^*$; u^2aub; u^2aub^{2*}
u, v	uv
u, v, a	uva; uva^2; $u^2va^\dagger$; $u^2va^{2\dagger}$; $u^2ava^{2\dagger}$
u, v, a, b	$uvab$; $uvba$; uva^2b^*; $uvba^{2*}$; uva^2b^2; uva^2ba^*; $u^2vab^\dagger$; $u^2avb^\dagger$; $u^2bva^{*\dagger}$

7. VOLUME CHANGES

Here we calculate the change of volume with deformation. A volume element $dV_0 \equiv dX_1dX_2dX_3$ at $\underset{\sim}{X} + \underset{\sim}{\Xi}$ with fixed $\underset{\sim}{\Xi}$ will be called a **material macrovolume element** and one $d\mathcal{V}_0 \equiv d\Xi_1d\Xi_2d\Xi_3$ with fixed $\underset{\sim}{X}$ will be called a **material minivolume element**. After deformation, dV_0 becomes dv and $d\mathcal{V}_0$ becomes $d\nu$ given

$$(7.1) \qquad dv = JdX_1dX_2dX_3$$

$$(7.2) \qquad d\nu = jd\Xi_1d\Xi_2d\Xi_3$$

where J and j are the jacobians of deformation with $\underset{\sim}{\Xi}$ and $\underset{\sim}{X}$ fixed respectively, i.e.,

$$(7.3) \qquad J \equiv \det (x_{k,K} + \chi_{kM,K}\Xi_M)$$

$$(7.4) \qquad j \equiv \det (\chi_{kK})$$

To obtain the ratios of deformed volume elements to those of undeformed ones, we need to calculate the jacobians J and j. Since the determinant of the product of matrices is equal to the product of the determinants of matrices, we have

$$J = \{\det [(x_{k,K} + \chi_{kM,K}\Xi_M)(x_{k,L} + \chi_{kN,L}\Xi_N)]\}^{1/2}$$

$$= \{\det (x_{k,K}x_{k,L} + x_{k,L}\chi_{kM,K}\Xi_M + x_{k,K}\chi_{kN,L}\Xi_N + \chi_{kM,K}\chi_{kN,L}\Xi_M\Xi_N)\}^{1/2}$$

Upon using (3.5) and (3.6) this becomes

$$(7.5) \qquad J = \{\det (C_{KL} + \Gamma_{LMK}\Xi_M + \Gamma_{KNL}\Xi_N + \Gamma_{PMK}\Gamma_{QNL}\overset{-1}{C}_{PQ}\Xi_M\Xi_N)\}^{1/2}$$

where

$$\overset{-1}{C}_{PQ} \equiv X_{P,k} X_{Q,k} \tag{7.6}$$

is the inverse tensor to C_{PQ}. In the calculation of the last term on the right-hand side, we employed

$$\chi_{kL,M} = \Gamma_{KLM} \chi_{K,k}$$

which follows from (3.7).

The above result (7.5) is valid for a general micromorphic material and can be further simplified. We are, however, interested only in a linear theory. In this case (7.5) can be approximated by

$$J \simeq \{\det [\delta_{KL} + 2E_{KL} + 2\Gamma_{(KL)}]\}^{1/2}$$

Expansion and linearization of this gives

$$J \simeq 1 + E_{KK} + \Gamma_{KK} \tag{7.7}$$

Thus the macrovolume change with fixed $\underset{\sim}{\Xi}$, to a linear approximation, is given by

$$\frac{dv}{dV_0} - 1 = \text{tr}\,\underset{\sim}{E} + (\underset{\sim}{\nabla} \times \underset{\sim}{\phi}) \cdot \underset{\sim}{\Xi} \tag{7.8}$$

where we used (4.9). Here the first term on the right-hand side is the classical expression of the dilatation, and the second term is the additional volume change due to microdeformation.

Similarly, we can calculate the minivolume element $d\nu$ by determining j given by (7.4). In this case

$$j = [\det (\chi_{kK} \chi_{kL})]^{1/2} \tag{7.9}$$

For the linear theory from (3.15) we have

$$\chi_{kK} \simeq (\delta_{MK} + \Phi_{MK})\delta_{KM}$$

Hence (7.9) to a linear approximation is

$$\text{(7.10)} \qquad j \simeq [\det(\delta_{KL} + \Phi_{KL} + \Phi_{LK})]^{1/2} = 1 + \Phi_{KK} + 0(\underset{\sim}{\Phi}^2)$$

For the micropolar theory $\Phi_{KK} = 0$ so that we have

$$\frac{d\nu}{dV_0} - 1 = 0(\Phi^2)$$

Hence, <u>in the linear micropolar theory there will be no minivolume change</u>. A minivolume (for $\underset{\sim}{X}$ fixed) rigidly rotates without altering its value.

8. COMPATIBILITY CONDITIONS

The strain tensor $e_{k\ell}$ and micropolar strain tensors $\varepsilon_{k\ell}$ and $\gamma_{k\ell m}$ are expressed in terms of the displacement field u_k and microrotation field ϕ_k by

$$e_{k\ell} = \frac{1}{2}(u_{k,\ell} + u_{\ell,k}) \tag{8.1}$$

$$\varepsilon_{k\ell} = u_{\ell,k} + \varepsilon_{\ell km}\phi_m \tag{8.2}$$

$$\gamma_{k\ell m} = \varepsilon_{k\ell n}\phi_{n,m} \tag{8.3}$$

Of these,(8.1) is the eulerian linear strain tensor known to us from the classical theory and (8.2) follows from (4.17) by use of (8.1) and (4.15) and (8.3) is identical to (4.18). When the six quantities u_k and ϕ_k are prescribed, these strain fields are determined uniquely through (8.1) to (8.3) by mere substitution. If instead the six strains $e_{k\ell}$, nine micro-strains $\varepsilon_{k\ell}$, and nine non-vanishing components of $\gamma_{k\ell m}$ are prescribed, then the determination of the displacement and microrotation fields requires the solution of twenty-four partial differential equations (8.1) to (8.3) for the six unknowns u_k and ϕ_k. Such a system is over-determined and restrictions must be imposed on $e_{k\ell}$, $\varepsilon_{k\ell}$, and $\gamma_{k\ell m}$. These conditions are known as the compatibility conditions. For the classical strain tensor $e_{k\ell}$, the compatibility conditions are (cf. Sokolnikoff [1956]):

$$e_{k\ell,mn} + e_{mn,k\ell} - e_{km,\ell n} - e_{\ell n,km} = 0 \tag{8.4}$$

If we notice that

$$e_{k} = \varepsilon_{(k\ell)} \equiv \frac{1}{2}(\varepsilon_{k\ell} + \varepsilon_{\ell k}) \tag{8.5}$$

then (8.4) can also be written as

$$\varepsilon_{(k\ell),mn} + \varepsilon_{(mn),k\ell} - \varepsilon_{(km),\ell n} - \varepsilon_{(\ell n),km} = 0 \tag{8.6}$$

An alternative derivation of (8.6) is instructive: To this end through (8.2) we calculate the displacement field u_k from

$$u_k = u_k^0 + \int_C (\varepsilon_{\ell k} + \varepsilon_{\ell km}\phi_m)dx_\ell \tag{8.7}$$

where u_k^0 is the value of u_k at one end x_m^0 of an open curve C in the body. Integrating by parts the second term under the integral, we also write

$$u_k = u_k^0 + \varepsilon_{\ell km}\phi_m(x_\ell - x_\ell^0) + \int_C [\varepsilon_{ik} + \varepsilon_{\ell km}(x_\ell - x_\ell^0)\phi_{m,i}]dx_i \tag{8.8}$$

For the displacement field u_k to be independent of the path C followed between the points x_m and x_m^0, the integrand must be expressible as a total differential, i.e.,

$$\varepsilon_{ik} - \varepsilon_{\ell km}(x_\ell - x_\ell^0)\phi_{m,i} = F_{k,i}$$

where $F_k(\underset{\sim}{x},t)$ are single-valued and possess continuous partial derivatives with respect to x_i, through second order. From this it follows that

$$F_{k,ij} = F_{k,ji}$$

Consequently

$$\varepsilon_{ik,j} - [\varepsilon_{\ell km}(x_\ell - x_\ell^0)\phi_{m,i}]_{,j} - \varepsilon_{jk,i} + [\varepsilon_{\ell km}(x_\ell - x_\ell^0)\phi_{m,j}]_{,i} = 0$$

Expanding the second term and using (8.3) we obtain

$$\varepsilon_{ik,j} - \varepsilon_{jk,i} + \gamma_{ikj} - \gamma_{jki} = 0 \tag{8.9}$$

These are the necessary and sufficient conditions for the displacement field u_k to be single-valued and continuous in a simply-connected region. Eliminating $\underset{\sim}{\gamma}$ from (8.9) by differentiating and using (8.2) and (8.3), we obtain (8.6) again.

A similar method can be applied to $\gamma_{k\ell m}$ as follows. First we solve for $\phi_{r,m}$ by multiplying both sides of (8.3) by $\varepsilon_{k\ell r}$. Hence

(8.10) $$\phi_{r,m} = \frac{1}{2} \varepsilon_{k\ell r} \gamma_{k\ell m}$$

Integration of (8.10) along a smooth open curve C gives

$$\phi_r = \phi_r^0 + \frac{1}{2} \int_C \varepsilon_{k\ell r} \gamma_{k\ell m} dx_m$$

The condition of single-valuedness for ϕ_r now reads

(8.11) $$\varepsilon_{k\ell r}(\gamma_{k\ell m,n} - \gamma_{k\ell n,m}) = 0$$

Thus we have proved the following.

Theorem: The necessary and sufficient conditions for the integrability of the system (8.1) to (8.3) for a simply-connected domain is the satisfaction of the compatibility conditions (8.4) and (8.9) and (8.11)*.

We note that the terms outside the line integral in (8.8) represent rigid body deformation.

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* Condition (8.9) was obtained by Sandru [1966]. For the general nonlinear theory of micromorphic materials, the proof was given by Eringen [1967].

9. SOME SPECIAL DEFORMATIONS

In this section we present a few special deformations for illustrative purposes.

(i) Rigid Deformation. The deformation of a body is called rigid if the distance between every pair of points $\underset{\sim}{X}^{(\alpha)}$ and $\underset{\sim}{Y}^{(\alpha)}$ in the body remains unchanged. From (3.33) it is clear that the necessary and sufficient condition for the rigid deformation of a linear micromorphic body is

(9.1) $$\underset{\sim}{E} = \underset{\sim}{\mathcal{E}} = \underset{\sim}{0} \quad , \quad \underset{\sim}{\Gamma} = \underset{\sim}{0}$$

Alternatively, in terms of the spatial measures of strains

(9.2) $$\underset{\sim}{e} = \underset{\sim}{\epsilon} = \underset{\sim}{0} \quad , \quad \underset{\sim}{\gamma} = \underset{\sim}{0}$$

For a micropolar body $E_{KL} = 0$ implies

(9.3) $$U_K = R_{KL}X_L + B_K$$

where R_{KL} is an arbitrary skew-symmetric tensor and B_K is an arbitrary vector, both of which are independent of $\underset{\sim}{X}$. The condition $\Gamma_{KL} = 0$ implies that Φ_K be independent of $\underset{\sim}{X}$. Finally $\mathcal{E}_{KL} = 0$ gives

(9.4) $$\Phi_K = R_K$$

where

(9.5) $$R_K = \frac{1}{2}\,\varepsilon_{KLM}R_{ML}$$

is a rotation vector independent of $\underset{\sim}{X}$.

(ii) Isochoric Deformations. The deformation will be called macroisochoric if the material macrovolume remains unchanged. It will be called miniisochoric if the minivolume is unchanged. The necessary and sufficient condition for macroisochoric deformations according to (7.7) or (7.8) is

(9.6) $$E_{KK} + \Gamma_{KK} = 0$$

The condition for miniisochoric deformation follows from (7.10). Since, for the linear micropolar bodies $\Phi_{KK} = 0$, we see that the <u>linear micropolar bodies undergo only miniisochoric deformation</u>. We note that, in general, this is not true for general micromorphic materials.

For the condition (9.6) to be valid for all $\underset{\sim}{\Xi}$, it is necessary and sufficient that

(9.7) $$E_{KK} = 0 \quad , \quad \Gamma_{KK} = 0$$

(iii) <u>Homogeneous Strain</u>. The state of strain in a body will be called <u>homogeneous</u> when the deformation is linear and homogeneous in the position vectors $\underset{\sim}{X}$ and $\underset{\sim}{\Xi}$ of the material points, i.e.,

(9.8) $$\underset{\sim}{x}^{(\alpha)} = \underset{\sim}{D}_K X_K + \underset{\sim}{\mathcal{D}}_K \Xi_K$$

where $\underset{\sim}{D}_K$ and $\underset{\sim}{\mathcal{D}}_K$ are constant vectors. In terms of the components of these vectors, (9.8) is equivalent to the system

(9.9) $$x_k = D_{kK} X_K$$

(9.10) $$\xi_k = \mathcal{D}_{kK} \Xi_K$$

where D_{kK} and $\mathcal{D}_{kK}$ are constants for static deformations and functions of time only for the dynamical motions and

(9.11) $$\underset{\sim}{D}_K = D_{kK} \underset{\sim}{i}_k \quad , \quad \underset{\sim}{\mathcal{D}}_K = \mathcal{D}_{kK} \underset{\sim}{i}_k$$

Equation (9.9) is the expression of homogeneous strain in classical continuum mechanics. The deformation described by this set carries straight lines into straight lines, ellipses into ellipses, ellipsoids into ellipsoids. The <u>microhomogeneous</u> deformation (9.10) is new, and it possesses the same kind of properties with respect to Ξ. For the micropolar body, $\mathcal{D}_{kK}$ can be replaced by a single vector $\mathcal{D}$ given through the solution of

$$\mathcal{D}_{kL} = \delta_{kL} - \varepsilon_{kLM}\mathcal{D}_M \tag{9.12}$$

namely

$$\mathcal{D}_M = -\frac{1}{2}\varepsilon_{kLM}\mathcal{D}_{kL} \tag{9.13}$$

Equation (9.12) results from equations (3.15) and (4.3) with $\Phi = \mathcal{D}$. Thus the microhomogeneous deformation can also be expressed by

$$\xi = \Xi - \Xi \times \mathcal{D} \tag{9.14}$$

where the vector $\mathcal{D}$ is independent of X and of course Ξ.

The material deformation tensors follow from (3.5) to (3.7) and (9.9) and (9.10) with $\mathcal{D}_{kK} = \chi_{kK}$. Thus

$$\begin{aligned} C_{KL} &= D_{kK}D_{kL} \\ \Psi_{KL} &= D_{kK}\mathcal{D}_{kL} \quad , \quad \Gamma_{KLM} = 0 \end{aligned} \tag{9.15}$$

It is now clear that the strain measures are homogeneous. For a micropolar solid we use (9.12) so that

$$\Psi_{KL} = D_{LK} - \varepsilon_{LMN}D_{NK}\mathcal{D}_M \tag{9.16}$$

where we put

$$D_{LK} \equiv \delta_{kL} D_{kK}$$

Below we give several special cases:

(iiia) Uniform Macrodilatation. In this case, $\underset{\sim}{D}$ is a diagonal matrix having the same entries, i.e.,

$$\underset{\sim}{D} = \begin{bmatrix} D & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{bmatrix} \quad , \quad 0 < D < \infty \tag{9.17}$$

The deformation tensors in this case take the forms

$$C_{KL} = D^2 \delta_{KL} \quad , \quad \Psi_{KL} = D\mathcal{D}_{kL}\delta_{kK} \quad , \quad \Gamma_{KLM} = 0 \tag{9.18}$$

The deformation carries a parallelepiped having edge vectors $\underset{\sim}{I}_{\underline{K}} dX_{\underline{K}}$ at $\underset{\sim}{X} + \underset{\sim}{\Xi}$ to one with edge vectors $DdX_{\underline{K}}\underset{\sim}{I}_{\underline{K}}$, Fig. 9.1. The ratio of the length of an edge to its original value (the macrostretch) $L_{(K)}$ is therefore given by

$$L_{(K)} = D$$

The angle between any two edge vectors of the deformed macroelement is 90°. The deformation carries a macrocube of unit volume to a macrocube of volume D^3. The microelement changes according to the values of $\mathcal{D}_K$. From (9.16), for this case, we have

$$\Psi_{KL} = D(\delta_{KL} - \varepsilon_{KLM}\mathcal{D}_M) \tag{9.19}$$

From this it is clear that

$$\Psi_{11} = \Psi_{22} = \Psi_{33} = D \quad , \quad \Psi_{KL} = -\varepsilon_{KLM} D\mathcal{D}_M \quad , \quad (K \neq L) \tag{9.20}$$

Thus, the microstretch $\ell_{(K)}$, the ratio of the edge vector of the deformed microelement to that of the deformed element, is given by

(9.21) $$\ell_{(K)} = D$$

Hence, the edges of the microelement are stretched the same amount as those of the macroelement. For D = 1 we have no macrodeformation or microdeformation. However, the microelement undergoes a rotation described by $\underset{\sim}{\mathcal{D}}$. For the general micromorphic materials, the situation is much more complicated and it is possible to have microdeformations even when the macrovolume remains unchanged. This situation is, of course, a familiar one in molecular theories of crystal lattices.

(iiib) Uniaxial Strain. Consider the homogeneous strain characterized by

(9.22) $$\underset{\sim}{D} = \begin{bmatrix} D & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad 0 < D < \infty$$

In this case, for a micropolar solid through (9.15) and (9.16), we get

(9.23) $$\underset{\sim}{C} = \begin{bmatrix} D^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad \underset{\sim}{\Psi} = \begin{bmatrix} D & -\mathcal{D}_3 D & \mathcal{D}_2 D \\ \mathcal{D}_3 D & 1 & -\mathcal{D}_1 \\ -\mathcal{D}_2 D & \mathcal{D}_1 & 1 \end{bmatrix}$$

Macrostretch $L_{(K)}$ and microstretch $\ell_{(K)}$ are

(9.24) $$\begin{aligned} L_{(1)} &= D \quad , \quad L_{(2)} = L_{(3)} = 1 \\ \ell_{(1)} &= D \quad , \quad \ell_{(2)} = \ell_{(3)} = 1 \end{aligned}$$

For $\mathcal{D}_1 = \mathcal{D}_2 = \mathcal{D}_3 = 0$, we have the classical uniaxial strain according to which a bar of length dX_1 after deformation becomes a bar of length DdX_1, Fig. 9.2. For non-vanishing $\mathcal{D}_K$ we see that the microelement is stretched by the amount D in the X_1 direction in addition to having a rotation. The geometry becomes particularly simple for the two-dimensional case for which we have

$$(9.25)\qquad \underset{\sim}{D} = \begin{bmatrix} D & 0 \\ 0 & 1 \end{bmatrix} \quad , \quad \underset{\sim}{C} = \begin{bmatrix} D^2 & 0 \\ 0 & 1 \end{bmatrix} \quad , \underset{\sim}{\Psi} = \begin{bmatrix} D & -\mathcal{D}_3 D \\ \mathcal{D}_3 D & 1 \end{bmatrix}$$

A sketch of the deformation is shown on Fig. 9.2. The macroelement OACB after deformation becomes OA'C'B elongated an amount $(D-1)dX_1$ along the X_1-axis with no change in the lateral directions. The microelement Cacb is stretched by the amount $Dd\Xi_1$ in X_1 direction becoming Da'c'b'. Afterwards, a rigid microrotation occurs about the X_3-axis at C. The final shape of Cacb is marked by Ca"c"b". For the general micromorphic materials, it is possible also to have microstretches independent of the macrodeformations.

Generally, when a bar is stretched in one direction without any constraints on its sides it will also change its lateral dimensions. This situation is characterized more realistically by

$$(9.26)\qquad \underset{\sim}{D} = \begin{bmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & D_3 \end{bmatrix}$$

In this case the state of strain is called <u>simple extension</u>. For simple extension we have

$$(9.27)\qquad \underset{\sim}{C} = \begin{bmatrix} D_1^2 & 0 & 0 \\ 0 & D_2^2 & 0 \\ 0 & 0 & D_3^2 \end{bmatrix} \quad , \quad \underset{\sim}{\Psi} = \begin{bmatrix} D_1 & -D_1\mathcal{D}_3 & D_1\mathcal{D}_2 \\ D_1\mathcal{D}_3 & D_2 & -D_2\mathcal{D}_1 \\ -D_1\mathcal{D}_2 & D_2\mathcal{D}_1 & D_3 \end{bmatrix}$$

From $(9.27)_1$ it is clear that a macroelement in the shape of a rectangular parallelepiped after deformation becomes another rectangular parallelepiped with its sides elongated proportionally to D_1, D_2, and D_3. A microelement in the shape of a rectangular parallelepiped changes its sides in the same proportions, however, it also rotates.

(iiic) Simple Shear. In classical continuum mechanics a homogeneous strain characterized by

$$\underset{\sim}{D} = \begin{bmatrix} 1 & S & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} , \quad -\infty < S < \infty \tag{9.28}$$

is called a simple shear. Here S is independent of X_K. The spatial position $x_k^{(\alpha)}$ of any material point $\underset{\sim}{X} + \underset{\sim}{\Xi}$ after deformation is given by

$$\begin{aligned} x_1^{(\alpha)} &= X_1 + SX_2 + \Xi_1 + \mathcal{D}_2\Xi_3 - \mathcal{D}_3\Xi_2 \\ x_2^{(\alpha)} &= X_2 + \Xi_2 + \mathcal{D}_3\Xi_1 - \mathcal{D}_1\Xi_3 \\ x_3^{(\alpha)} &= X_3 + \Xi_3 + \mathcal{D}_1\Xi_2 - \mathcal{D}_2\Xi_1 \end{aligned} \tag{9.29}$$

In the case of $\mathcal{D}_K = 0$, simple shear rotates X_1 = const. planes rigidly about their lines of intersection with X_2 = 0-plane, by an amount equal to the angle of shear γ given by

$$\gamma = \text{arc tan } S \tag{9.30}$$

The X_2 = const. and X_3 = const. planes are unchanged (cf. Eringen [1962, Art. 15]). The deformation tensors $\underset{\sim}{C}$, $\underset{\sim}{\Psi}$, and $\underset{\sim}{\Gamma}$ are given by

(9.31)
$$\underset{\sim}{C} = \begin{bmatrix} 1 & S & 0 \\ S & 1+S^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad \underset{\sim}{\Psi} = \begin{bmatrix} 1 & -\mathcal{D}_3 & \mathcal{D}_2 \\ (S+1)\mathcal{D}_3 & 1-S\mathcal{D}_3 & S\mathcal{D}_2-\mathcal{D}_1 \\ -\mathcal{D}_2 & \mathcal{D}_1 & 1 \end{bmatrix}$$

$$\underset{\sim}{\Gamma} = \underset{\sim}{0}$$

From (9.31) we see that even when there is no macroshear, that is when $S = 0$, we shall have a microrotation prescribed by the microdeformation tensor

$$\underset{\sim}{\Psi} = \begin{bmatrix} 1 & -\mathcal{D}_3 & \mathcal{D}_2 \\ \mathcal{D}_3 & 1 & -\mathcal{D}_1 \\ -\mathcal{D}_2 & \mathcal{D}_1 & 1 \end{bmatrix} \tag{9.32}$$

The picture for a plane microdeformation, in this case, is similar to the one described by Fig. 9.2.

(iv) Plane Strain. In classical continuum mechanics when the deformation of a body is identical in a family of parallel planes and vanishes in the directions of their common normal, we say that a state of plane strain exists. This plane strain is thus characterized by

$$x_k = x_k(X_1, X_2) \quad , \quad (k = 1,2) \quad , \quad x_3 = X_3 \tag{9.33}$$

For the micropolar deformations, we define the state of plane-micropolar strain similarly. Using (4.11) and setting $\Phi_1 \equiv \Phi_2 \equiv 0$, $\Phi_3 \equiv \Phi$ we have

$$\begin{aligned} \xi_1 &= \Xi_1 - \Phi(X_1,X_2)\Xi_2 \\ \xi_2 &= \Xi_2 + \Phi(X_1,X_2)\Xi_3 \\ \xi_3 &= \Xi_3 \end{aligned} \tag{9.34}$$

Thus, in a plane macrostrain and microstrain the deformation is fully prescribed when the two unknown displacements

$$(9.35)\qquad \begin{aligned} U_1(X_1,X_2) &\equiv x_1 - X_2 \\ U_2(X_1,X_2) &\equiv x_2 - X_2 \end{aligned}$$

and the microdisplacement $\Phi(X_1,X_2)$ is determined. The field equations should, therefore, consist of three partial differential equations replacing the two equations of the classical theory. This, as we shall see, is the case (cf. Arts. 25 and 26).

The spatial deformation tensors for the plane strain are

$$(9.36)\qquad \underset{\sim}{C} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} , \quad \underset{\sim}{\Psi} = \begin{bmatrix} \Psi_{11} & \Psi_{12} & 0 \\ \Psi_{21} & \Psi_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Gamma_{12M} = -\Gamma_{21M} = -\Phi_{,M} \quad , \quad (M = 1,2)$$

Equations (3.27) to (3.29) and (4.3) provide the relations between deformations, tensors, and strains (or displacement vectors):

$$(9.37)\qquad \begin{aligned} C_{KL} &= \delta_{KL} + 2E_{KL} = \delta_{KL} + U_{K,L} + U_{L,K} \\ \Psi_{KL} &= \mathcal{E}_{KL} + \delta_{KL} = \delta_{KL} - \varepsilon_{KLM}\Phi_M + U_{L,K} \\ \Gamma_{KLM} &= -\varepsilon_{KLN}\Phi_{N,M} \end{aligned}$$

Thus, for the strain tensors E_{KL} and $\mathcal{E}_{KL}$ we have

$$(9.38)\qquad \underset{\sim}{E} = \begin{bmatrix} E_{11} & E_{12} & 0 \\ E_{21} & E_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} , \quad \underset{\sim}{\mathcal{E}} = \begin{bmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} & 0 \\ \mathcal{E}_{21} & \mathcal{E}_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

when

$$E_{11} = \frac{\partial U_1}{\partial X_1} \quad , \quad E_{12} = E_{21} = \frac{1}{2}\left(\frac{\partial U_1}{\partial X_2} + \frac{\partial U_2}{\partial X_1}\right) \quad , \quad E_{22} = \frac{\partial U_2}{\partial X_2}$$

$$\mathcal{E}_{11} = \frac{\partial U_1}{\partial X_1} \quad , \quad \mathcal{E}_{12} = -\phi + \frac{\partial U_2}{\partial X_1} \tag{9.39}$$

$$\mathcal{E}_{21} = \phi + \frac{\partial U_1}{\partial X_2} \quad , \quad \mathcal{E}_{22} = \frac{\partial U_2}{\partial X_2}$$

10. MOTION, MICROMOTION, MATERIAL DERIVATIVE OF TENSORS

A material point at $\underset{\sim}{X} + \underset{\sim}{\Xi}$ at time $t = 0$ is carried to a spatial point $\underset{\sim}{x} + \underset{\sim}{\xi}^{(\alpha)}$ at time t. The motion of this material point in a body is described by one parameter family of transformations.

$$\underset{\sim}{x}^{(\alpha)} = \underset{\sim}{x}(\underset{\sim}{X},t) + \underset{\sim}{\xi}(\underset{\sim}{X},\underset{\sim}{\Xi},t) \tag{10.1}$$

where $\underset{\sim}{x}(\underset{\sim}{X},t)$ is the place occupied by the center of mass $\underset{\sim}{X}$ of a macrovolume element $dV + dS$ at time t and $\underset{\sim}{\xi}(\underset{\sim}{X},\underset{\sim}{\Xi},t)$ is the relative position of the point $\underset{\sim}{X} + \underset{\sim}{\Xi}$ at time t with respect to the center of mass. For micromorphic bodies we have (3.9) or

$$\underset{\sim}{x}(\underset{\sim}{X},t) = \underset{\sim}{X} + \underset{\sim}{u}(\underset{\sim}{X},t) \tag{10.2}$$

for $\underset{\sim}{x}(\underset{\sim}{X},t)$ and (2.9) for $\underset{\sim}{\xi}(\underset{\sim}{X},\underset{\sim}{\Xi},t)$. For micropolar bodies, (10.2) remains valid but (2.9) is replaced by (4.11), i.e.,

$$\underset{\sim}{\xi}(\underset{\sim}{X},\underset{\sim}{\Xi},t) = \underset{\sim}{\Xi} - \underset{\sim}{\Xi} \times \underset{\sim}{\phi}(\underset{\sim}{X},t) \tag{10.3}$$

In these expressions $\underset{\sim}{U}(X,t)$ and $\underset{\sim}{\phi}(X,t)$ are respectively the macrodisplacement vector and microdisplacement vector. The parameter t is real representing time.

According to the axiom of continuity and indestructibility of matter, the inverse motions $\underset{\sim}{X}(\underset{\sim}{x},t)$ and $\underset{\sim}{\Xi}(\underset{\sim}{x},\underset{\sim}{\xi},t)$ are assumed to exist. Thus we may also write

$$\underset{\sim}{X}^{(\alpha)} = \underset{\sim}{X}(\underset{\sim}{x},t) + \underset{\sim}{\Xi}(\underset{\sim}{x},\underset{\sim}{\ },t) \tag{10.4}$$

where $\underset{\sim}{\Xi}$ is given by (2.13) for micromorphic bodies. We also have

$$\underset{\sim}{X}(\underset{\sim}{x},t) = \underset{\sim}{x} - \underset{\sim}{u}(\underset{\sim}{x},t) \tag{10.5}$$

However, for micropolar bodies (2.13) is replaced by (4.12), i.e.,

$$\underset{\sim}{\Xi} = \underset{\sim}{\xi} + \underset{\sim}{\xi} \times \underset{\sim}{\phi}(\underset{\sim}{x},t) \tag{10.6}$$

For displacement vectors, we employ the same symbol $\underset{\sim}{u}$ and $\underset{\sim}{\phi}$ both in material and spatial descriptions. However, in the former case $\underset{\sim}{u}$ and $\underset{\sim}{\phi}$ are assumed to be functions of $\underset{\sim}{X}$ and t, and in the latter case functions of $\underset{\sim}{x}$ and t since we may substitute $\underset{\sim}{X} = X(\underset{\sim}{x},t)$ for $\underset{\sim}{X}$ in $\underset{\sim}{u}(\underset{\sim}{X},t)$ and $\underset{\sim}{\phi}(\underset{\sim}{X},t)$ to pass from material description to spatial descriptions. Single-valued inverses $\underset{\sim}{X}(\underset{\sim}{x},t)$ to $\underset{\sim}{x}(\underset{\sim}{X},t)$ and $\underset{\sim}{\Xi}(\underset{\sim}{X},\underset{\sim}{\xi},t)$ to $\underset{\sim}{\xi} = \underset{\sim}{\xi}(\underset{\sim}{X},\underset{\sim}{\Xi},t)$ are assumed to exist at a neighborhood of $\underset{\sim}{X}$ at all times except possibly some singular points, lines, and surfaces in the body. A sufficient condition for this is the continuity of partial derivatives of these functions with respect to X_K and

$$\det\,(x_{k,K}) \neq 0 \qquad \det\,(\phi_{k,K}) \neq 0 \tag{10.7}$$

in some neighborhood of X_K at all times. We assume that such is the case.

In the kinematics of continuous media, the time rates of vectors and tensors associated with material points play an important role.

<u>Def. 1. The material derivative of any tensor is defined as the partial derivative of that tensor with respect to time with the material coordinates X_K and Ξ_K held constant.</u> The material derivative is indicated either by placing a dot on the letters or by D/Dt. Thus, for example,

$$\begin{aligned}
\dot{F}_K(\underset{\sim}{X},t) &\equiv \frac{DF_K}{Dt} = \left.\frac{\partial F_K(\underset{\sim}{X},t)}{\partial t}\right|_{\underset{\sim}{X}} \\
\dot{f}_k(\underset{\sim}{x},t) &\equiv \frac{Df_k}{Dt} = \left.\frac{\partial f_k(\underset{\sim}{x},t)}{\partial t}\right|_{\underset{\sim}{X}} \\
\dot{\xi}_k &\equiv \frac{D\xi_k}{Dt} = \left.\frac{\partial \xi_k}{\partial t}\right|_{\underset{\sim}{X},\underset{\sim}{\Xi}} = \dot{\underset{\sim}{\chi}}_K(\underset{\sim}{X},t)\,\Xi_K
\end{aligned} \tag{10.8}$$

where subscripts attached to a bar indicate that those variables are held constant during the differentiation. Since through the centroidal motion we have

(10.9) $$x_k = x_k(\underset{\sim}{X},t) \quad \text{or} \quad X_K = X_K(\underset{\sim}{x},t)$$

we have

$$f_k(\underset{\sim}{x},t) = f_k(\underset{\sim}{x}(\underset{\sim}{X},t),t)$$

so that

$$\dot{f}_k = \left.\frac{\partial f_k}{\partial t}\right|_{\underset{\sim}{x}} + \left.\frac{\partial f_k}{\partial x_\ell}\right|_t \left.\frac{\partial x_\ell(\underset{\sim}{X},t)}{\partial t}\right|_{\underset{\sim}{X}}$$

In short, without ambiguity we write

(10.10) $$\dot{f}_k \equiv \frac{Df_k}{Dt} = \frac{\partial f_k}{\partial t} + f_{k,\ell}\dot{x}_\ell$$

The first term on the extreme right of this equation represents the time rate of change that occurs at a place $\underset{\sim}{x}$ at time t. The second group of terms is known as the <u>convective change</u>. These arise from the motion of the material point $\underset{\sim}{X}$ through the place $\underset{\sim}{x}$.

In the case when the tensors involve the macromotion in the material differentiation, we also consider the relative location vector Ξ_K held constant, cf. $(10.8)_3$. Another example is provided by differentiation of (4.11).

(10.11)

$$\dot{\underset{\sim}{\xi}} = -\underset{\sim}{\Xi} \times \dot{\underset{\sim}{\Phi}} \quad \text{or} \quad \dot{\xi}_k = -\varepsilon_{kLM}\Xi_L \frac{\partial \Phi_M(\underset{\sim}{X},t)}{\partial t}$$

11. VELOCITY, ACCELERATION, MICROROTATION, SPIN

Def. 1. Velocity is the time rate of change of the position vector of a material point. Thus

$$(11.1) \qquad \underset{\sim}{v} \equiv \dot{\underset{\sim}{x}}(\underset{\sim}{X},t) \quad \text{or} \quad v_k = \dot{x}_k$$

where $\underset{\sim}{X}$ is held constant. Since $\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{X},t)$ we have

$$\underset{\sim}{v} = \frac{\partial \underset{\sim}{x}(\underset{\sim}{X},t)}{\partial t} = \dot{\underset{\sim}{x}}(\underset{\sim}{X},t)$$

Note that in a body with microstructure, $\underset{\sim}{X}$ is the position of the center of mass of a macrovolume element and it may or may not be actually occupied by a material point. Nevertheless, in defining the time rates of vectors and tensors associated with the body, we refer to $\underset{\sim}{X}$ as the material point.

Upon replacing $\underset{\sim}{X}$ by $(10.9)_2$ we also write

$$(11.2) \qquad \underset{\sim}{v} = \dot{\underset{\sim}{x}}(\underset{\sim}{X}(\underset{\sim}{x},t),t) = \dot{x}_k(x,t)\underset{\sim}{i}_k = v_k(\underset{\sim}{x},t)\underset{\sim}{i}_k$$

where $\underset{\sim}{i}_k$ are the spatial rectangular unit base vectors. This equation defines the velocity field v_k at a spatial point $\underset{\sim}{x}$ at time t. This is the eulerian concept of the velocity field which is prominent in hydrodynamics. For the lagrangian viewpoint, we express the velocity vector in the material frame of reference X_K. Thus

$$(11.3) \qquad \underset{\sim}{v} = V_K(\underset{\sim}{X},t)\underset{\sim}{I}_K \quad , \quad V_K \equiv \frac{\partial x_k(\underset{\sim}{X},t)}{\partial t}\,\delta_{kK}$$

Def. 2. Acceleration is the time rate of change of the velocity vector of a material point. Thus

$$(11.4) \qquad \underset{\sim}{a} \equiv \dot{\underset{\sim}{v}} \quad \text{or} \quad a_k = \dot{v}_k = \frac{Dv_k}{Dt}$$

For the lagrangian viewpoint we have

$$a = A_K(X,t) I_K \quad , \quad A_K \equiv \frac{\partial V_K(X,t)}{\partial t} \tag{11.5}$$

and for the eulerian viewpoint

$$a_k \equiv \frac{Dv_k}{Dt} = \frac{\partial v_k}{\partial t} + v_{k,\ell} v_\ell \tag{11.6}$$

Here we notice the appearance of the convective terms $v_{k,\ell}v_\ell$

The velocity v and acceleration a defined above are the kinematical quantities describing the motion of the center of mass X in a macromaterial element V + S. We now proceed to obtain the relative velocity and acceleration of a material point $X + \Xi$ with respect to the center of mass X. For these, we take the time rates of the relative motion for a micropolar body given by (2.9), namely,

$$\xi = \chi_K(X,t) \Xi_K \tag{11.7}$$

Thus

$$\begin{aligned} \dot{\xi} &= \dot{\chi}_K(X,t) \Xi_K \\ \ddot{\xi} &= \ddot{\chi}_K(X,t) \Xi_K \end{aligned} \tag{11.8}$$

Alternative expressions are obtained by replacing Ξ_K by its expression

$$\Xi_K = X_{Kk}(x,t) \xi_k \tag{11.9}$$

Thus, for example,

$$\dot{\xi} = \nu_k(x,t) \xi_k \quad \text{or} \quad \dot{\xi}_\ell = \nu_{\ell k} \xi_k \tag{11.10}$$

where

$$\nu_k(x,t) \equiv \dot{\chi}_K(X,t)X_{Kk}(x,t) \quad , \quad \nu_{\ell k} \equiv \dot{\chi}_{\ell K}X_{Kk} \tag{11.11}$$

It is understood that X appearing in the argument of $\dot{\chi}_K$ is also replaced by $(10.9)_2$.

Def. 3. The three vectors ν_k defined by (11.11) are called the gyration vectors, and their components $\nu_{\ell k}$ form the gyration tensor.

When the gyration tensor is given, we can calculate the eulerian microvelocity $\dot{\xi}$ by (11.10). For the microacceleration in a similar fashion, we obtain

$$\ddot{\xi} = \dot{\nu}_k\xi_k + \nu_k\dot{\xi}_k = \dot{\nu}_k\xi_k + \nu_k\nu_{k\ell}\xi_\ell$$

where we used $(11.10)_2$. Thus

$$\ddot{\xi} = \alpha_k(x,t)\xi_k \quad \text{or} \quad \ddot{\xi}_k = \alpha_{k\ell}\xi_\ell \tag{11.12}$$

where

$$\alpha_k(x,t) \equiv \dot{\nu}_k + \nu_m\nu_{mk} \quad \text{or} \quad \alpha_{\ell k} \equiv \dot{\nu}_{\ell k} + \nu_{\ell m}\nu_{mk} \tag{11.13}$$

Def. 4. The three vectors $\alpha_k(x,t)$ defined by (11.13) are called the spin tensor.

The total velocity $v^{(\alpha)}$ and acceleration $a^{(\alpha)}$ of a material point $X + \Xi$ can now be calculated by

$$v^{(\alpha)} = \dot{x} + \dot{\xi} = v + \nu_k\xi_k \tag{11.14}$$

$$a^{(\alpha)} = a + \alpha = \dot{v} + \alpha_k\xi_k \tag{11.15}$$

For a micropolar body, these expressions are modified by use of (4.4) and (4.14). Thus, for example,

$$\nu_{k\ell} = -\varepsilon_{k\ell M}\dot{\Phi}_M + \varepsilon_{kKM}\varepsilon_{K\ell m}\dot{\Phi}_M\phi_m \tag{11.16}$$

For the __linear theory__ this is simplified to

$$\nu_{k\ell} \simeq -\varepsilon_{k\ell m}\dot{\phi}_m \tag{11.17}$$

Upon introducing an axial vector ν_k, called microgyration vector, by

$$\nu_k = \frac{1}{2}\varepsilon_{k\ell m}\nu_{m\ell} \quad , \quad \nu_{k\ell} = -\varepsilon_{k\ell m}\nu_m \tag{11.18}$$

we see that

$$\nu_k = \dot{\phi}_k \tag{11.19}$$

and $(11.10)_1$ now reads

$$\dot{\underset{\sim}{\xi}} = -\underset{\sim}{\xi} \times \underset{\sim}{\nu} \tag{11.20}$$

Similarly, one can calculate $\ddot{\underset{\sim}{\xi}}$. An alternative approach that may be instructive is through taking the time rate of (11.20)

$$\ddot{\underset{\sim}{\xi}} = -\underset{\sim}{\xi} \times \dot{\underset{\sim}{\nu}} - \dot{\underset{\sim}{\xi}} \times \underset{\sim}{\nu} = -\underset{\sim}{\xi} \times \dot{\underset{\sim}{\nu}} + (\underset{\sim}{\xi} \times \underset{\sim}{\nu}) \times \underset{\sim}{\nu} \tag{11.21}$$

If we recall a vector identity

$$(\underset{\sim}{a} \times \underset{\sim}{b}) \times \underset{\sim}{c} = (\underset{\sim}{a} \cdot \underset{\sim}{c})\underset{\sim}{b} - (\underset{\sim}{b} \cdot \underset{\sim}{c})\underset{\sim}{a} \tag{11.22}$$

the above expression can be written as

$$\ddot{\underset{\sim}{\xi}} = -\underset{\sim}{\xi} \times \dot{\underset{\sim}{\nu}} + (\underset{\sim}{\xi} \cdot \underset{\sim}{\nu})\underset{\sim}{\nu} - (\underset{\sim}{\nu} \cdot \underset{\sim}{\nu})\underset{\sim}{\xi} \tag{11.23}$$

whose component form is

(11.24) $$\ddot{\xi}_k = \alpha_{k\ell}\xi_\ell$$

where

(11.25) $$\alpha_{k\ell} \equiv -\varepsilon_{k\ell m}\dot{\nu}_m + \nu_k\nu_\ell - \nu_m\nu_m\delta_{k\ell}$$

It can be seen that this is identical to $(11.13)_2$.

The total velocity and acceleration vectors of a material point $\underset{\sim}{X} + \underset{\sim}{\Xi}$ in a micropolar body can now be expressed, respectively, by

(11.26) $$\underset{\sim}{v}^{(\alpha)} = \dot{\underset{\sim}{x}}^{(\alpha)} = \dot{\underset{\sim}{x}} + \dot{\underset{\sim}{\xi}} = \underset{\sim}{\nu} - \underset{\sim}{\xi} \times \underset{\sim}{\nu}$$

(11.27) $$\underset{\sim}{a}^{(\alpha)} = \dot{\underset{\sim}{v}}^{(\alpha)} = \underset{\sim}{a} + \ddot{\underset{\sim}{\xi}} = \dot{\underset{\sim}{v}} - \underset{\sim}{\xi} \times \dot{\underset{\sim}{\nu}} + (\underset{\sim}{\xi} \times \underset{\sim}{\nu}) \times \underset{\sim}{\nu}$$

Here $\underset{\sim}{v}$ and $\dot{\underset{\sim}{v}}$ refer to the centroidal point of the macrovolume element, and the remaining terms on the extreme rights of (11.26) and (11.27) are the relative velocity and acceleration about the centroid. Equation (11.27) can be linearized further by dropping the triple vector product term on the extreme right.

12. MATERIAL DERIVATIVE OF ARC LENGTH

In continuum mechanics, the time rates of arc length, elements of surface, and volume in the deformed configuration are often required. Here we prepare the groundwork for this, while at the same time introducing certain new concepts essential to the study of motion.

Fundamental Lemma 1. The material derivative of the displacement gradient is given by

$$\frac{D}{Dt}(x_{k,K}) = \overline{\dot{x_{k,K}}} = v_{k,\ell}x_{\ell,K} \tag{12.1}$$

The proof of this is immediate since D/Dt and $\partial/\partial X_K$ can be exchanged, i.e.

$$\frac{D}{Dt}[x_{k,K}(\underset{\sim}{X},t)] = (\frac{Dx_k}{Dt})_{,K} = \dot{x}_{k,K} = v_{k,\ell}x_{\ell,K}$$

where we used $\dot{x}_k = v_k(\underset{\sim}{x},t)$ and the chain rule of differentiation. Another useful expression that follows from (12.1) by multiplying it by dX_K is

$$\overline{\dot{dx_k}} = v_{k,\ell}dx_\ell \tag{12.2}$$

A corollary to Fundamental Lemma 1 is

$$\frac{D}{Dt}(X_{K,k}) = -X_{K,\ell}v_{\ell,k} \tag{12.3}$$

which is proved by differentiating $x_{k,K}X_{K,\ell} = \delta_{k\ell}$. Thus

$$\overline{\dot{x_{k,K}}}\, X_{K,\ell} + x_{k,K}\,\overline{\dot{X_{K,\ell}}} = 0$$

Now multiply this by $X_{L,k}$. This gives (12.3).

<u>Theorem 1</u>. <u>The material derivative of the square of the arc length ds^2 is given by</u>

$$\overline{\dot{ds^2}} = 2d_{k\ell}dx_k dx_\ell \tag{12.4}$$

where

$$d_{k\ell} \equiv v_{(k,\ell)} \equiv \frac{1}{2}(v_{k,\ell} + v_{\ell,k}) \tag{12.5}$$

is called the <u>deformation rate tensor</u>.

To prove (12.5) we take the time rate of ds^2:

$$\overline{\dot{ds^2}} = \frac{D}{Dt}(dx_k dx_k) = 2\,\overline{\dot{dx_k}}\,dx_k = 2v_{k,\ell}dx_k dx_\ell$$

$$= (v_{k,\ell} + v_{\ell,k})dx_k dx_\ell$$

Hence the proof.

In the material description, (12.4) can be written as

$$\overline{\dot{ds^2}} = 2d_{k\ell}x_{k,K}x_{\ell,L}dX_K dX_L$$

If we use

$$ds^2 = C_{KL}dX_K dX_L = (\delta_{KL} + 2E_{KL})dX_K dX_L$$

then

$$\overline{\dot{ds^2}} = \dot{C}_{KL}dX_K dX_L = 2\dot{E}_{KL}dX_K dX_L \tag{12.6}$$

By comparing this with the foregoing expression, and since $d_{k\ell}$ and C_{KL} and E_{KL} are symmetric tensors, we find that

$$\dot{C}_{KL} = 2\dot{E}_{KL} = 2d_{k\ell}x_{k,K}x_{\ell,L} \tag{12.7}$$

This is the material derivative of the lagrangian strain measures $\underset{\sim}{C}$ and $\underset{\sim}{E}$.

When $\underset{\sim}{d} = \underset{\sim}{0}$ we have $D(ds^2)/Dt = 0$. Conversely, when for arbitrary $d\underset{\sim}{x}$, $D(ds^2)/Dt = 0$ we must have $\underset{\sim}{d} = \underset{\sim}{0}$. Hence we have

Theorem 2 (Killing). The necessary and sufficient condition for the macromotion $\underset{\sim}{x}(\underset{\sim}{X},t)$ to be rigid is $\underset{\sim}{d} = \underset{\sim}{0}$.

Note that macrorigid motion does not imply microrigid motion. As we shall see below, the microelements may undergo non-rigid motions even though macroelements may be moving rigidly.

Fundamental Lemma 2. The material derivative of the micromotion is given by

$$\dot{\chi}_{kK}(\underset{\sim}{X},t) = \nu_{k\ell}\chi_{\ell K} \tag{12.8}$$

This result follows from $(11.11)_2$ by multiplying it by χ_{kL} and using $(2.11)_2$.

A corollary to (12.8) is

$$\dot{\overline{\mathcal{X}_{Kk}}} = -\mathcal{X}_{K\ell}\nu_{\ell k} \tag{12.9}$$

which is obtained by taking the material time rate of (2.11) and multiplying the result by $\mathcal{X}_{Lk}$.

Theorem 3. The material derivative of the microdisplacement gradient $\chi_{kK,L}$ is given by

$$\frac{D}{Dt}(\chi_{kK,L}) \equiv \dot{\chi}_{kK,L} = \nu_{k\ell}\chi_{\ell K,L} + \nu_{k\ell,m}\chi_{\ell K}x_{m,L} \tag{12.10}$$

To prove this, we take the partial derivative of (12.8) with respect to X_L and exchange D/Dt and $\partial/\partial X_L$ since this is permissible.

A <u>corollary</u> to this theorem is

$$\dot{\overline{d\xi_k}} = \nu_{km} d\xi_m + \nu_{k\ell,m}\xi_\ell dx_m \tag{12.11}$$

which is proved by taking the material derivative of

$$d\xi_k = \chi_{kK,L}\Xi_K dX_L + \chi_{kK} d\Xi_K \tag{12.12}$$

and using (12.8), (12.10), and $\xi_k = \chi_{kK}\Xi_K$.

<u>Theorem 4. The material derivative of the square of the arc length $(ds^{(\alpha)})^2$ is given by</u> [1]

$$\begin{aligned}\frac{D}{Dt}[(ds^{(\alpha)})^2] &= [v_{k,\ell} + v_{\ell,k} + (\nu_{kr,\ell} + \nu_{\ell r,k})\xi_r]dx_k dx_\ell \\ &+ 2(v_{\ell,k} + \nu_{k\ell} + \nu_{\ell r,k}\xi_r)dx_k d\xi_\ell + (\nu_{k\ell} + \nu_{\ell k})d\xi_k d\xi_\ell\end{aligned} \tag{12.13}$$

To prove this, we take the material derivative of

$$(ds^{(\alpha)})^2 = dx_k dx_k + 2dx_k d\xi_k + d\xi_k d\xi_k \tag{12.14}$$

and use (12.2) and (12.11).

If we now introduce the <u>microdeformation rates</u>

$$b_{k\ell} \equiv \nu_{k\ell} + v_{\ell,k} \tag{12.15}$$

$$a_{k\ell m} \equiv \nu_{k\ell,m} \ , \quad a_{k\ell}^{(\alpha)} \equiv \nu_{kr,\ell}\xi_r^{(\alpha)} \tag{12.16}$$

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[1] Eringen [1964c]

equation (12.13) can be expressed in the form

$$\frac{D}{Dt}[(ds^{(\alpha)})^2] = 2[d_{k\ell} + a_{(k\ell)}]dx_k dx_\ell + 2(b_{k\ell} + a_{\ell k})dx_k d\xi_\ell$$

(12.17)

$$+ 2[b_{(k\ell)} - d_{k\ell}]d\xi_k d\xi_\ell$$

In a region of a micromorphic body when

(12.18) $$\underset{\sim}{d} = \underset{\sim}{0} \quad , \quad \underset{\sim}{b} = \underset{\sim}{0}$$

we have the general solution

(12.19) $$v_k = \omega_{k\ell}x_\ell + \dot{b}_k \quad , \quad \nu_{k\ell} = \omega_{k\ell}$$

where $\omega_{k\ell}$ is an angular velocity and $\dot{b}_k$ is a velocity, both of which are independent of $\underset{\sim}{x}$, and

(12.20) $$\omega_{k\ell} + \omega_{\ell k} = 0$$

Upon substituting (12.19) into (12.16), we see that $a_{k\ell m} = 0$. Conversely, we can show that the vanishing $\underset{\sim}{d}$ and $\underset{\sim}{b}$ is also necessary for $D[(ds^{(\alpha)})^2]/Dt = 0$. Hence we have proved

Theorem 5. The necessary and sufficient conditions for a micromorphic body to undergo microrigid motion are

(12.21) $$\underset{\sim}{b} = \underset{\sim}{d} = \underset{\sim}{0}$$

This theorem replaces the well-known Killing's theorem (Theorem 2 above) for micromorphic bodies.

For linear micropolar bodies, considerable simplification is achieved in the foregoing results. To this end we recall (4.4) and $(11.18)_2$, namely,

(12.22) $$\chi_{kK} = \delta_{kK} - \epsilon_{kKM}\Phi_M \quad , \quad \nu_{k\ell} = -\epsilon_{k\ell m}\nu_m$$

Substitution of these into (12.8) and (12.10) and linearization gives, respectively,

$$\dot{x}_{kK} \simeq -\varepsilon_{kKm}\nu_m \quad , \quad \dot{x}_{kK,L} \simeq -\varepsilon_{kKm}\nu_{m,r}\delta_{rL} \tag{12.23}$$

The expression of (12.11) for a linear micropolar body is

$$\dot{\overline{d\xi_k}} = -\varepsilon_{k\ell m}d\xi_\ell\nu_m - \varepsilon_{k\ell m}\nu_{m,r}\xi_\ell dx_r \tag{12.24}$$

or in vector notation

$$\dot{\overline{d\underset{\sim}{\xi}}} = -d\underset{\sim}{\xi} \times \underset{\sim}{\nu} - \underset{\sim}{\xi} \times \underset{\sim}{\nu}_{,r}dx_r \tag{12.25}$$

The material derivative of the square of the arc length for this case is

$$\frac{D}{Dt}[(ds^{(\alpha)})^2] = 2[d_{k\ell} + a_{(k\ell)}]dx_k dx_\ell + 2(b_{k\ell} + a_{\ell k})dx_k d\xi_\ell \tag{12.26}$$

where

$$b_{k\ell} \equiv v_{\ell,k} - \varepsilon_{k\ell m}\nu_m \quad , \quad a_{k\ell}^{(\alpha)} \equiv -\varepsilon_{krm}\nu_{m,\ell}\xi_r \tag{12.27}$$

13. RATES OF STRAIN MEASURES

<u>Def. 1. The time rates of various strain measures are the same as their material derivatives.</u> Thus, for example,

$$\dot{E}_{KL} \equiv \frac{DE_{KL}}{Dt} \quad , \quad \dot{e}_{k\ell} \equiv \frac{De_{k\ell}}{Dt}$$

$$\dot{\mathcal{E}}_{KL} \equiv \frac{D\mathcal{E}_{KL}}{Dt} \quad , \quad \dot{\varepsilon}_{k\ell} \equiv \frac{D\varepsilon_{k\ell}}{Dt} \tag{13.1}$$

$$\dot{\Gamma}_{KLM} \equiv \frac{D\Gamma_{KLM}}{Dt} \quad , \quad \dot{\gamma}_{k\ell m} \equiv \frac{D\gamma_{k\ell m}}{Dt}$$

We now proceed to give explicit expressions for these quantities.

<u>Theorem 1. The lagrangian strain rates are given by</u>

$$\dot{E}_{KL} = d_{k\ell} x_{k,K} x_{\ell,L} \tag{13.2}$$

$$\dot{\mathcal{E}}_{KL} = b_{k\ell} x_{k,K} X_{\ell L} \tag{13.3}$$

$$\dot{\Gamma}_{KLM} = b_{k\ell} x_{k,K} X_{\ell L,M} + a_{k\ell m} x_{k,K} X_{\ell L} x_{m,M} \tag{13.4}$$

The proof of (13.2) has already been given in Art. 12. To prove (13.3), we calculate the material derivative of

$$\mathcal{E}_{KL} = \Psi_{KL} - \delta_{KL} = x_{k,K} X_{kL} - \delta_{KL} \tag{13.5}$$

Hence

$$\dot{\mathcal{E}}_{KL} = \dot{\Psi}_{KL} = \dot{\overline{x_{k,K}}}\, X_{kL} + x_{k,K}\, \dot{\overline{X_{kL}}}$$

Upon using (12.1) and (12.8) we obtain (13.3). The proof of (13.4) is constructed similarly by taking the time rate of

$$\Gamma_{KLM} \equiv x_{k,K}\chi_{kL,M} \tag{13.6}$$

and using (12.1) and (12.10).

Theorem 2. The eulerian strain rates are given by

$$\dot{e}_{k\ell} = d_{k\ell} - (e_{km}v_{m,\ell} + e_{m\ell}v_{m,k}) \tag{13.7}$$

$$\dot{\varepsilon}_{k\ell} = b_{k\ell} - (\varepsilon_{km}\nu_{m\ell} + \varepsilon_{m\ell}v_{m,k}) \tag{13.8}$$

$$\dot{\gamma}_{k\ell m} = -a_{k\ell m} + \varepsilon_{kr}a_{r\ell m} - (\gamma_{k\ell r}v_{r,m} + \gamma_{krm}\nu_{r\ell} + \gamma_{r\ell m}v_{r,k}) \tag{13.9}$$

The proofs of these are somewhat lengthy and will not be given here. They are obtained by differentiating the expressions for the strains and using various results obtained in Art. 12. For the proof of (13.7) see Eringen [1962, Art. 22], and for (13.8) and (13.9) see Eringen [1967].

Equations (13.2) to (13.4) and (13.7) to (13.9) indicate that the eulerian strain rates are not the same as the deformation rates. If at time t the medium is unstrained and the motion is just beginning, we can set $\underset{\sim}{x} = \underset{\sim}{X}$ and $\underset{\sim}{\xi} = \underset{\sim}{\Xi}$ so that

$$\begin{aligned} \dot{E}_{KL}(\underset{\sim}{X},0) &= d_{k\ell}\delta_{kK}\delta_{\ell L} \\ \dot{\mathcal{E}}_{KL}(\underset{\sim}{X},0) &= b_{k\ell}\delta_{kK}\delta_{\ell L} \\ \dot{\Gamma}_{KLM}(\underset{\sim}{X},0) &= a_{k\ell m}\delta_{kK}\delta_{\ell L}\delta_{mM} \end{aligned} \tag{13.10}$$

and

$$\begin{aligned} \dot{e}_{k\ell}(\underset{\sim}{x},0) &= d_{k\ell} \\ \dot{\varepsilon}_{k\ell}(\underset{\sim}{x},0) &= b_{k\ell} \\ \dot{\gamma}_{k\ell m}(\underset{\sim}{x},0) &= -a_{k\ell m} \end{aligned} \tag{13.11}$$

For the infinitesimal deformation theory, the terms enclosed in parentheses on the right-hand side of (13.7) to (13.9) can be neglected. Therefore, in this case (13.11) should be valid approximately for all times, i.e.,

$$
\begin{aligned}
&\dot{e}_{k\ell}(\underset{\sim}{x},t) \simeq d_{k\ell} \\
(13.12) \qquad &\dot{\varepsilon}_{k\ell}(\underset{\sim}{x},t) \simeq b_{k\ell} \\
&\dot{\gamma}_{k\ell m}(\underset{\sim}{x},t) \simeq -a_{k\ell m}
\end{aligned}
$$

14. EXTERNAL AND INTERNAL LOADS

A material body subject to the external and internal forces undergoes a deformation. These forces may be of mechanical, electrical, chemical, and other origin. Here we are only concerned with the mechanical forces. In the particle mechanics of Newton, the force $\underset{\sim}{F}$ acting on a particle is considered to be a function of the position of the particle $\underset{\sim}{x}$, its velocity $\underset{\sim}{v}$, and time t, i.e.,

$$\underset{\sim}{F} = \underset{\sim}{F}(\underset{\sim}{x},\underset{\sim}{v},t) \tag{14.1}$$

When we have a collection of particles, then for each particle we may write

$$\underset{\sim}{F}_\alpha = \underset{\sim}{F}_\alpha(\underset{\sim}{x}_\alpha,\underset{\sim}{v}_\alpha,t) \quad , \quad (\alpha = 1,2,\ldots) \tag{14.2}$$

In a volume element ΔV of a continuum, we have a large number of particles interconnected with such forces. If the particles of a continuum are not free to move independently, then the interparticle forces are balanced among themselves in pairs. This then places restrictions on (14.2) so that the number of independent forces is much smaller than the free collection of particles. In conformity with these restrictions and with the basic postulates of continuum mechanics, the forces acting on a body are resolved into a resultant force $\underset{\sim}{F}$ and a resultant couple $\underset{\sim}{M}$ given by

$$\underset{\sim}{F} = \sum_\alpha \underset{\sim}{F}_\alpha \quad , \quad \underset{\sim}{M} = \sum_\alpha \underset{\sim}{x}_\alpha \times \underset{\sim}{F}_\alpha \tag{14.3}$$

The first of these equations gives the vector sum of all forces acting on each material point with position $\underset{\sim}{x}_\alpha$, and the second gives the vector sum of the moments of these forces about a point which constitutes the origin of $\underset{\sim}{x}_\alpha$. In a continuum, the force field is usually considered to be continuous and (14.3) may be replaced by

$$\underset{\sim}{F} = \int_V d\underset{\sim}{F} \quad , \quad \underset{\sim}{M} = \int_V (\underset{\sim}{x} \times d\underset{\sim}{F} + d\underset{\sim}{M}) \tag{14.4}$$

where $d\underset{\sim}{F}$ is the force density at a point $\underset{\sim}{x}$ and $d\underset{\sim}{M}$ is a couple density. This latter terms arises in classical mechanics from coupling various particles in the form of doublets or rigid blocks so that some of the applied forces on particles produce a couple also. This physical picture can be used in the construction of the theories of micromechanics.

From a continuum viewpoint, whatever the origin may be, the forces and couples may be divided into three categories.

a. Extrinsic Body Loads. These are the forces and couples that arise from the external effects. They act on the mass points of the body. They appear in the form of body forces and body couples per unit mass of the body. The force of gravity is an example of a body force, and an electromagnetic moment in a polarized medium is an example of a body couple. A body couple can also arise from the uneven distribution of the mass among microvolume elements, Fig. 14.1 and Fig. 14.2.

b. Extrinsic Surface Loads (Contact Loads). These loads arise from the action of one body on another through the contacting surfaces. At a small macrosurface they are equipollent to a force and a couple. Thus, for example, forces acting on a macrosurface of Fig. 14.3 are equipollent to a force and a couple placed at the centroid of the macrosurface element Δa, Fig. 14.4.

When a macrovolume Δv is allowed to tend to dv, in general the body couple $\underset{\sim}{L}$ vanishes since the moment arms of forces tend to zero while the forces are assumed to remain bounded. Similarly, when Δa is allowed to tend to da, the surface couple $\underset{\sim}{M}$ will approach to zero. This is the classical picture in continuum mechanics.

Because of the granular nature of the bodies, the mathematical limits dv and da for the surface and volume elements are approximations which may not be admissible for various physical phenomena in which the applied loads produce effects with some typical lengths (e.g., wave length) that are comparable to distances and sizes of the microelements. In such situations, Δv and Δa are not infinitesimals, and the granular nature of the bodies must be taken into account in some form. This then requires that we consider the existence of both forces and couplesfor macrovolume and surface elements.

c. Internal Loads. Internal loads arise from the mutual action of pairs of particles that are located inside the body. According to Newton's third law, the interparticle forces cancel each other so that the resultant force is zero.

In continuum mechanics, the internal effects are found by isolating a small macroelement from the body and considering the effect of the rest of the body as forces and couples on the surface of the macroelement as illustrated in Figs. 14.1 to 14.4. Internal forces give rise to the stress and couple stress hypotheses,as we shall see below.

Let the surface force per unit area at a point $\underset{\sim}{x}$ on the surface of a body having exterior normal $\underset{\sim}{n}$ be denoted by $\underset{\sim}{t}_{(\underset{\sim}{n})}$, and the surface couple per unit area by $\underset{\sim}{m}_{(\underset{\sim}{n})}$. Let the body force and body couple per unit mass at an interior point of the body be respectively represented by $\underset{\sim}{f}$ and $\underset{\sim}{\ell}$. The total force $\underset{\sim}{F}$ and the total couple $\underset{\sim}{M}$ about a point 0 acting on the body can be calculated by, Fig. 14.5

$$\underset{\sim}{F} = \oint_S \underset{\sim}{t}_{(\underset{\sim}{n})} da + \int_V \rho \underset{\sim}{f} dv \tag{14.5}$$

$$\underset{\sim}{M} = \oint_S [\underset{\sim}{m}_{(\underset{\sim}{n})} + \underset{\sim}{x} \times \underset{\sim}{t}_{(\underset{\sim}{n})}] da + \int_V \rho(\underset{\sim}{\ell} + \underset{\sim}{x} \times \underset{\sim}{f}) dv \tag{14.6}$$

Concentrated loads are imagined as resulting from a limiting process in which the surface loads or body loads are distributed over a very small region.

15. MECHANICAL BALANCE LAWS

The mechanical balance laws - conservation of mass, balance of momentum, and balance of moment of momentum - are obtained by a process of averaging applied to a macrovolume element containing N microelements for which the classical balance laws are postulated to be valid. Each of the microelements is considered to have a uniform mass density.

I. Principle of Conservation of Mass. The total mass of each microelement remains constant during any deformation. Thus, [illegible] $\rho_0^{(\alpha)}$ and $\rho^{(\alpha)}$ respectively denote the mass densities of a microelement α before and after deformation, and $\Delta V_0^{(\alpha)}$ and $\Delta v^{(\alpha)}$ their volume, Fig. 15.1, then

$$\rho_0^{(\alpha)} \Delta V_0^{(\alpha)} = \rho^{(\alpha)} \Delta v^{(\alpha)} \quad , \quad (\alpha \text{ not summed}) \tag{15.1}$$

The total mass of a macrovolume before and after deformations is respectively given by

$$\rho_0 \Delta V_0 \equiv \sum_{\alpha=1}^{N} \rho_0^{(\alpha)} \Delta V_0^{(\alpha)} \tag{15.2}$$

$$\rho \Delta \equiv \sum_{\alpha=1}^{N} \rho^{(\alpha)} \Delta v^{(\alpha)} \tag{15.3}$$

These equations in effect define the mass densities $\rho_0(\underset{\sim}{X})$ and $\rho(\underset{\sim}{x},t)$ of the undeformed and deformed macrovolume elements. In view of (15.1) we see that

$$\rho_0 \Delta V_0 = \rho \Delta v \tag{15.4}$$

If we let ΔV_0 and ΔV approach their limiting values dV_0 and dV, then

$$\rho_0 dV_0 = \rho dv$$

(15.5)

or

$$\frac{\rho_0}{\rho} = \frac{dv}{dV_0} \equiv J = \det (x_{k,K})$$

which are the equivalent expressions of the principle of conservation of mass for the macrovolume element.

In Section 2 we said that $\underset{\sim}{X}$ is the position vector of the center of mass of a macroelement. Accordingly,

$$\sum_{\alpha}\rho_0^{(\alpha)}\underset{\sim}{\Xi}^{(\alpha)}\Delta V^{(\alpha)} = 0$$

Upon using (15.1) and (2.14) this gives

$$\underset{\sim}{X}_k\sum_{\alpha}\rho^{(\alpha)}\xi_k^{(\alpha)}\Delta v^{(\alpha)} = 0$$

Since $\underset{\sim}{X}_k \neq 0$, this shows that the position vector $\underset{\sim}{x}$ is the center of mass of the deformed macrovolume. Consequently,

<u>Theorem 1</u>. <u>The motion carries the center of mass of the undeformed macrovolume to the center of mass of the deformed macrovolume</u>[1].

Nect, we calculate the second moments

$$\rho_0 I_{KL}\Delta V \equiv \sum_{\alpha}\rho_0^{(\alpha)}\Xi_K^{(\alpha)}\Xi_L^{(\alpha)}\Delta V_0^{(\alpha)} \tag{15.6}$$

upon substituting (15.1) and (2.13), this may be written as (cf. Eringen [1964c])

$$I_{KL} = i_{k\ell}X_{Kk}X_L \tag{15.7}$$

where

$$\rho i_{k\ell}\Delta v \equiv \sum_{\alpha}\rho^{(\alpha)}\xi^{(\alpha)}\xi^{(\alpha)}\Delta v^{(\alpha)} \tag{15.8}$$

Quantities I_{KL} and $i_{k\ell}$ are respectively called the material and spatial microinertia tensors. Equations (15.7) may be stated as

---

[1] Eringen [1964c]

<u>Theorem 2</u>. <u>The microinertia is conserved</u>, i.e.,

$$\frac{D}{Dt}(i_{k\ell}X_{Kk}X_{L\ell}) = 0 \tag{15.9}$$

Using (12.9), this may also be expressed as (Eringen [1964c])

$$\frac{\partial i_{k\ell}}{\partial t} + i_{k\ell,m}v_m - i_{km}\nu_{\ell m} - i_{m\ell}\nu_{km} = 0$$

In a micropolar continuum, a combination of $i_{k\ell}$ and I_{KL} occur more frequently. These are

$$\begin{aligned} J_{KL} &\equiv I_{MM}\delta_{KL} - I_{KL} \\ j_{k\ell} &\equiv i_{mm}\delta_{k\ell} - i_{k\ell} \end{aligned} \tag{15.10}$$

These tensors are identical to the inertia tensor encountered in rigid-body dynamics.

Upon linearization and using (4.14), (15.7), and (15.11), we get

$$J_{KL} \simeq j_{k\ell}\delta_{Kk}\delta_{L\ell} \tag{15.11}$$

Global balance equations for mass and microinertia are obtained by integrating (15.4) and (15.7) over the volume of the body. Thus

$$\int_V \rho_0 \, dV_0 = \int_{\mathcal{V}} \rho \, dv \tag{15.12}$$

$$\int_V \rho_0 I_{KL} \, dV_0 = \int_{\mathcal{V}} \rho i_{k\ell} X_{Kk} X_{L\ell} \, dv \tag{15.13}$$

where V is the undeformed volume and $\mathcal{V}$ is the deformed material volume.

II. <u>Principle of Balance of Momentum</u>. <u>The time rate of change of momentum is equal to the sum of all forces acting on a body</u>.

The mechanical momentum of a microelement $\Delta v^{(\alpha)}$ is the product of its mass with the velocity, namely, $\rho^{(\alpha)}\underset{\sim}{v}^{(\alpha)}\Delta v^{(\alpha)}$. The total momentum of a macroelement is the vector sum of the micromomenta of its microelements. For a micropolar body we have

$$\Delta \underset{\sim}{p} = \sum_\alpha \rho^{(\alpha)}\underset{\sim}{v}^{(\alpha)}\Delta v^{(\alpha)} = \sum_\alpha \rho^{(\alpha)}(\underset{\sim}{v} + \dot{\underset{\sim}{\xi}})\Delta v^{(\alpha)}$$

$$= \underset{\sim}{v}\sum_\alpha \rho^{(\alpha)}\Delta v^{(\alpha)} + \underset{\sim}{\nu} \times \sum_\alpha \rho^{(\alpha)}\underset{\sim}{\xi}\Delta v^{(\alpha)}$$

The last term vanishes and in the limit we write

$$d\underset{\sim}{p} = \rho\underset{\sim}{v}dv$$

The total momentum of the body is therefore given by

$$\underset{\sim}{p} = \int_V \rho\underset{\sim}{v}dv \tag{15.14}$$

The principle of balance of momentum is expressed by

$$\frac{d}{dt}\int_V \rho\underset{\sim}{v}dv = \oint_S \underset{\sim}{t}_{(\underset{\sim}{n})} \cdot d\underset{\sim}{a} + \int_V \rho\underset{\sim}{f}dv \tag{15.15}$$

Here $\underset{\sim}{t}_{(\underset{\sim}{n})}$ is the surface traction per unit area acting on the surface of the body S with an outward directed normal $\underset{\sim}{n}$ so that the surface integral is the vector sum of all forces acting on S. The vector sum of the body forces is given by the volume integral on the right. Equation (15.15) is none other than that given in classical continuum mechanics.

III. <u>Balance of Moment of Momentum</u>. <u>The time rate of change of moment of momentum about a point is equal to the sum of all couples and the moment of all forces about that point</u>.

The mechanical moment of momentum of a microelement is defined as the moment of its momentum, namely,

$$\underset{\sim}{x}^{(\alpha)} \times \rho^{(\alpha)} \underset{\sim}{v}^{(\alpha)} \Delta v^{(\alpha)}$$

The total moment of momentum of a macroelement is calculated by

$$d\underset{\sim}{M} = \sum_{\alpha} \underset{\sim}{x}^{(\alpha)} \times \rho^{(\alpha)} \underset{\sim}{v}^{(\alpha)} \Delta v^{(\alpha)}$$

$$= \sum_{\alpha} (\underset{\sim}{x} + \underset{\sim}{\xi}) \times \rho^{(\alpha)} (\underset{\sim}{v} + \dot{\underset{\sim}{\xi}}) \Delta v^{(\alpha)}$$

Upon carrying out the multiplication, we get

$$\Delta \underset{\sim}{m} = \underset{\sim}{x} \times \underset{\sim}{v} \sum_{\alpha} \rho^{(\alpha)} \Delta v^{(\alpha)} + \sum_{\alpha} \underset{\sim}{\xi} \times \rho^{(\alpha)} \dot{\underset{\sim}{\xi}} \Delta v^{(\alpha)}$$

$$+ \underset{\sim}{x} \times \sum_{\alpha} \rho^{(\alpha)} \dot{\underset{\sim}{\xi}} \Delta v^{(\alpha)} - \underset{\sim}{v} \times \sum_{\alpha} \rho^{(\alpha)} \underset{\sim}{\xi} \Delta v^{(\alpha)}$$

The last two summations vanish since $\underset{\sim}{\xi}$ is measured from the center of mass of the deformed macroelement. Upon carrying the expression $\dot{\underset{\sim}{\xi}}$ from (11.20), in the limit we may write

(15.16) $$d\underset{\sim}{m} = \rho \underset{\sim}{x} \times \underset{\sim}{v} dv + \rho \underset{\sim}{\sigma} dv$$

where

(15.17) $$\rho \underset{\sim}{\sigma} \Delta v \equiv \sum_{\alpha} \rho^{(\alpha)} \underset{\sim}{\xi} \times (\underset{\sim}{\nu} \times \underset{\sim}{\xi}) \Delta v^{(\alpha)}$$

is called the <u>intrinsic</u> <u>spin</u>. In component form this reads

(15.18) $$\sigma_{\ell} = j_{k\ell} \nu_k$$

where we used (15.8) and $(15.11)_2$ after expanding the triple product.

The total moment of momentum of a macroelement, therefore, is the vector sum of its angular momentum and the intrinsic spin. The total moment of momentum of a micropolar body is now calculated by

$$\underset{\sim}{M} = \int_{V} (\underset{\sim}{x} \times \rho \underset{\sim}{v} + \rho \underset{\sim}{\sigma}) dv \tag{15.19}$$

The principle of moment of momentum is expressed by

$$\begin{aligned} \frac{d}{dt} \int_{V} (\underset{\sim}{x} \times \rho \underset{\sim}{v} + \rho \underset{\sim}{\sigma}) dv = & \oint_{S} (\underset{\sim}{x} \times \underset{\sim}{t}_{(\underset{\sim}{n})} + \underset{\sim}{m}_{(\underset{\sim}{n})}) da \\ & + \int_{V} \rho (\underset{\sim}{\ell} + \underset{\sim}{x} \times \underset{\sim}{f}) dv \end{aligned} \tag{15.20}$$

The right-hand side gives the sum of all moments about the origin as in (14.6).

16. STRESS AND COUPLE STRESS

The state of internal loads and their connections to surface loads may be found by applying the principles of global balance of momenta to small regions fully and partially contained in the body, Fig. 16.1. To this end, we first consider a small macrovolume, v + s, fully contained in the body. At a point $\underset{\sim}{x}$ of s, the effect of the remainder of the body is equipollent to a surface force per unit area, $\underset{\sim}{t}_{(\underset{\sim}{n})}$, called the <u>stress vector</u>, and a couple per unit area, $\underset{\sim}{m}_{(\underset{\sim}{n})}$, called the <u>couple stress vector</u>. These loads depend on the position $\underset{\sim}{x}$, time t, and the orientation of the surface s at $\underset{\sim}{x}$ which is described by the exterior normal $\underset{\sim}{n}$ to s at $\underset{\sim}{x}$. This latter dependence can be found explicitly by applying the mechanical principles of momenta to a region v + s adjacent to the surface of the body. This approach does in fact also provide the connection of surface loads to the internal loads. Consider a small tetrahedron with three faces taken as the coordinate surfaces and the fourth face being a part of the surface of the body, Fig. 16.2. We denote the stress vectors on any coordinate surface x_k = const. by $-\underset{\sim}{t}_k$ and on s by $\underset{\sim}{t}_{(\underset{\sim}{n})}$. The equation of balance of momentum (15.15) can be applied on this tetrahedron. Using the mean-value theorem to estimate the volume and surface integrals, we write

$$\frac{d}{dt}(\overset{*}{\rho}\overset{*}{\underset{\sim}{v}}\Delta v) = \overset{*}{\underset{\sim}{t}}_{(\underset{\sim}{n})}\Delta a - \overset{*}{\underset{\sim}{t}}_k \Delta a_k + \rho \overset{*}{f}\Delta v$$

where the quantities marked with asterisks are the values of those without asterisks at some points of v + s. The volume is denoted by Δv and the surface areas by Δa_k and Δa. The mass is conserved so that

$$\frac{d}{dt}(\rho dv) = 0 \qquad (16.1)$$

Upon dividing both sides of the foregoing equation by Δv and letting Δa and Δv approach zero, we see that $\Delta v/\Delta a \to 0$ and we obtain

$$\underset{\sim}{t}_{(\underset{\sim}{n})} da = \underset{\sim}{t}_k da_k \tag{16.2}$$

The four surfaces of the tetrahedron form a closed surface, therefore, the limit of the sum of area vectors da_k must add up to da. Hence

$$d\underset{\sim}{a} = \underset{\sim}{n} da = da_k \underset{\sim}{i}_k \tag{16.3}$$

From this we get

$$da_k = n_k da \tag{16.4}$$

Substituting this into (16.2) we get

$$\underset{\sim}{t}_{(\underset{\sim}{n})} = \underset{\sim}{t}_k n_k \tag{16.5}$$

where $\underset{\sim}{t}_k$ is independent of $\underset{\sim}{n}$. Thus we found that the stress vector $\underset{\sim}{t}_{(\underset{\sim}{n})}$ is a linear function of $\underset{\sim}{n}$. At two sides of a surface, $\underset{\sim}{n}$ changes sign. From (16.5) we therefore see that

$$\underset{\sim}{t}_{(-\underset{\sim}{n})} = -\underset{\sim}{t}_{(\underset{\sim}{n})} \tag{16.6}$$

which proves that <u>the stress vectors on opposite sides of the same surface at a given point are equal in magnitude and opposite in sign</u>.

The application of the above method, with the use of the equation of balance of moment of momentum (15.20) leads to

$$\underset{\sim}{m}_{(\underset{\sim}{n})} = \underset{\sim}{m}_k n_k \tag{16.7}$$

$$\underset{\sim}{m}_{(-\underset{\sim}{n})} = -\underset{\sim}{m}_{(\underset{\sim}{n})} \tag{16.8}$$

The concepts of stress tensor $t_{k\ell}$ and couple stress tensor $m_{k\ell}$ now follow from the decompositions

$$t_k = t_{k\ell} i_\ell \tag{16.9}$$

$$m_k = m_{k\ell} i_\ell \tag{16.10}$$

Thus $t_{k\ell}$ is the ℓ^{th} component of the stress vector t_k which acts on the surface x_k = const. and $m_{k\ell}$ is the ℓ^{th} component of the couple stress vector which acts on the same surface. The positive directions of $t_{k\ell}$ and those of $m_{k\ell}$ are shown on Figs. 16.3 and 16.4 respectively. We use double-headed arrows for $m_{k\ell}$.

From (16.5), (16.7), (16.9), and (16.10) it follows that

$$t_{(n)} = t_{k\ell} n_k i_\ell \tag{16.11}$$

$$m_{(n)} = m_{k\ell} n_k i_\ell \tag{16.12}$$

It is thus clear that the moment vectors for the couple stress have the identical sign convention to those of the stress vectors. The plane of each couple is of course perpendicular to the couple vector, and the direction is as described by the right-hand screw rule.

The expanded form of the components of $t_{(n)}$ and $m_{(n)}$ in rectangular coordinates are

$$\begin{aligned} t_{(n)x} &= t_{xx} n_x + t_{yx} n_y + t_{zx} n_z \\ t_{(n)y} &= t_{xy} n_x + t_{yy} n_y + t_{zy} n_z \\ t_{(n)z} &= t_{xz} n_x + t_{yz} n_y + t_{zz} n_z \end{aligned} \tag{16.13}$$

$$
\begin{aligned}
m_{(\underset{\sim}{n})x} &= m_{xx}n_x + m_{yx}n_y + m_{zx}n_z \\
m_{(\underset{\sim}{n})y} &= m_{xy}n_x + m_{yy}n_y + m_{zy}n_z \qquad (16.14) \\
m_{(\underset{\sim}{n})z} &= m_{xz}n_x + m_{yz}n_y + m_{zz}n_z
\end{aligned}
$$

17. LOCAL BALANCE LAWS

Local balance laws are obtained by postulating that the global balance laws are valid for every part of the body. For the conservation of mass, we convert the volume integral over $\mathcal{V}$ to V. Thus

$$\int_V (\rho_0 - \rho J)dv = 0 \tag{17.1}$$

where

$$J \equiv \det (x_{k,K}) \tag{17.2}$$

is the jacobian of the transformation. Postulating that (17.1) is valid for every part of the body, we obtain the equation of local mass conservation.

$$\rho_0/\rho = J \tag{17.3}$$

Another form often used in hydrodynamics is obtained from this by taking the material derivative of (17.3). Thus

$$\dot{\rho}J + \rho\dot{J} = 0$$

and we can show that (Eringen [1962, Art. 19])

$$\dot{J} - Jv_{k,k} = 0$$

Consequently

$$\dot{\rho} + \rho v_{k,k} = 0$$

or since

$$\dot{\rho} = \frac{\partial\rho}{\partial t} + \rho_{,k}v_k$$

this reads

$$\frac{\partial\rho}{\partial t} + (\rho v_k)_{,k} = 0 \tag{17.4}$$

This is the celebrated equation of continuity. One may, of course, equally employ (17.3) in place of (17.5). Alternative forms which follows from (15.4) are

$$\rho_0 dV_0 = \rho dv \quad , \quad \dot{\overline{\rho dv}} = 0 \tag{17.5}$$

An expanded form of (17.4) in rectangular coordinates is

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} = 0 \tag{17.6}$$

where (v_x, v_y, v_z) are the rectangular components of the velocity field.

Equations of local balance for the microinertia are already given by (15.7) and (15.9). For the linear theory using (15.11) we get

$$\frac{Dj_{k\ell}}{Dt} = 0 \tag{17.7}$$

The local balance of momenta follows from (15.15) and (15.20). Upon carrying out the indicated differentiation and using $(17.5)_2$, and writing $\underset{\sim}{a} = \dot{\underset{\sim}{v}}$, $\underset{\sim}{v} = \dot{\underset{\sim}{x}}$, we see that

$$\int_{\underset{\sim}{V}} \rho \underset{\sim}{a} dv = \oint_{\underset{\sim}{S}} \underset{\sim}{t}_{(n)} da + \int_{\underset{\sim}{V}} \rho \underset{\sim}{f} dv \tag{17.8}$$

$$\int_{\underset{\sim}{V}} (\underset{\sim}{x} \times \rho \underset{\sim}{a} + \rho \dot{\underset{\sim}{\sigma}}) dv = \oint_{\underset{\sim}{S}} (\underset{\sim}{x} \times \underset{\sim}{t}_{(n)} + \underset{\sim}{m}_{(n)}) da + \int_{\underset{\sim}{V}} \rho (\underset{\sim}{\ell} + \underset{\sim}{x} \times \underset{\sim}{f}) dv \tag{17.9}$$

These are other forms of the global balance of momenta. We now take $V + \underset{\sim}{S}$ to be a small internal portion $v + s$ of the body. Substituting from (16.5) and (16.7) we write

$$\int_v \rho \underset{\sim}{a} dv = \oint_s \underset{\sim}{t}_k n_k da + \int_v \rho \underset{\sim}{f} dv \tag{17.10}$$

$$\int_v \rho(\underset{\sim}{x} \times \underset{\sim}{a} + \dot{\underset{\sim}{\sigma}}) dv = \oint_s (\underset{\sim}{x} \times \underset{\sim}{t}_k + \underset{\sim}{m}_k) n_k da + \int_v \rho(\underset{\sim}{\ell} + \underset{\sim}{x} \times \underset{\sim}{f}) dv \tag{17.11}$$

In rectangular coordinates, the Green-Gauss theorem is expressed as

$$\oint_s \underset{\sim}{g}_k n_k da = \int_v \underset{\sim}{g}_{k,k} dv \tag{17.12}$$

If we now apply this theorem to (17.10) and (17.11) to convert the surface integrals to volume integrals, we obtain

$$\int_v [\underset{\sim}{t}_{k,k} + \rho(\underset{\sim}{f} - \underset{\sim}{a})] dv = \underset{\sim}{0} \tag{17.13}$$

$$\int_v [\underset{\sim}{m}_{k,k} + \underset{\sim}{i}_k \times \underset{\sim}{t}_k + \rho(\underset{\sim}{\ell} - \dot{\underset{\sim}{\sigma}})] dv + \int_v \underset{\sim}{x} \times [\underset{\sim}{t}_{k,k} + \rho(\underset{\sim}{f} - \underset{\sim}{a})] dv = \underset{\sim}{0} \tag{17.14}$$

For these equations to be valid for any arbitrary volume v in the body, the necessary and sufficient condition is the vanishing of the integrands. Hence

$$\underset{\sim}{t}_{k,k} + \rho(\underset{\sim}{f} - \dot{\underset{\sim}{v}}) = 0 \tag{17.15}$$

$$\underset{\sim}{m}_{k,k} + \underset{\sim}{i}_k \times \underset{\sim}{t}_k + \rho(\underset{\sim}{\ell} - \dot{\underset{\sim}{\sigma}}) = 0 \tag{17.16}$$

Note that the second integrand in (17.14) vanishes by virtue of (17.15). These equations are the expressions of the local balance of momenta. They are identical to those given in Eringen [1962, eqs. (32.7) and (32.8)] with the exception of the spin inertia term $\rho\dot{\underset{\sim}{\sigma}}$. This term arises from the postulate of an independent microrotation. In fact, without such an internal degree of freedom, the existence of $\underset{\sim}{m}_k$ and $\underset{\sim}{\ell}$ is questionable. Upon substituting (16.9) and (16.10) into (17.15) and (17.16), we obtain the component form of these equations, namely,

$$t_{\ell k,\ell} + \rho(f_k - \dot{v}_k) = 0 \tag{17.17}$$

$$m_{\ell k,\ell} + \varepsilon_{kmn} t_{mn} + \rho(\ell_k - \dot{\sigma}_k) = 0 \tag{17.18}$$

These are the <u>first and second laws of motion</u> of Cauchy which express the local balance of momenta for micropolar bodies[1]. When the body is nonpolar, that is, when $\underset{\sim}{\sigma} = \underset{\sim}{m}_k = \underset{\sim}{\ell} = \underset{\sim}{0}$, then (17.18) gives the classical result

$$t_{k\ell} = t_{\ell k} \tag{17.19}$$

which expresses the symmetry of the stress tensor. For micropolar bodies, we see that the stress is in general nonsymmetrical and the new set of differential equations (17.18) must be employed replacing (17.19).

In rectangular coordinates, the expanded expressions of (17.17) and (17.18) are recorded below

$$\begin{aligned}
&\frac{\partial t_{xx}}{\partial x} + \frac{\partial t_{yx}}{\partial y} + \frac{\partial t_{zx}}{\partial z} + \rho(f_x - \dot{v}_x) = 0 \\
&\frac{\partial t_{xy}}{\partial x} + \frac{\partial t_{yy}}{\partial y} + \frac{\partial t_{zy}}{\partial z} + \rho(f_y - \dot{v}_y) = 0 \\
&\frac{\partial t_{xz}}{\partial x} + \frac{\partial t_{yz}}{\partial y} + \frac{\partial t_{zz}}{\partial z} + \rho(f_z - \dot{v}_z) = 0
\end{aligned} \tag{17.20}$$

---

[1] If we disregard the relation of σ_k to ν_k given by (15.18), these balance laws are valid for the nonlinear theory, and (17.17) and (17.18) are exact expressions, cf. Eringen and Suhubi [1964a & b].

$$\frac{\partial m_{xx}}{\partial x} + \frac{\partial m_{yx}}{\partial y} + \frac{\partial m_{zx}}{\partial z} + t_{yz} - t_{zy} + \rho(\ell_x - \dot{\sigma}_x) = 0$$

$$\frac{\partial m_{xy}}{\partial x} + \frac{\partial m_{yy}}{\partial y} + \frac{\partial m_{zy}}{\partial z} + t_{zx} - t_{xz} + \rho(\ell y - \dot{\sigma} y) = 0 \tag{17.21}$$

$$\frac{\partial m_{xz}}{\partial x} + \frac{\partial m_{yz}}{\partial y} + \frac{\partial m_{zz}}{\partial z} + t_{xy} - t_{yx} + \rho(\ell_z - \dot{\sigma}_z) = 0$$

18. CONSERVATION OF ENERGY

An important axiom of thermomechanics is the principle of conservation of energy which may be stated as follows:

IV. Principle of Conservation of Energy. The time rate of change of the sum of the kinetic energy and the internal energy is equal to the sum of the mechanical energy, heat energy, and other energies. Here we exclude chemical and electrical energies so that we may express this law mathematically as

$$\dot{K} + \dot{E} = W + Q \tag{18.1}$$

Here K, E, W and Q are, respectively, the kinetic energy, the internal energy, the work of applied loads per unit time, and the heat energy. For a micropolar continuum, these quantities may be expressed as:

$$K = \frac{1}{2}\int_V \rho(v_k v_k + j_{k\ell}\nu_k\nu_\ell)dv \tag{18.2}$$

$$E = \int_V \rho\varepsilon dv \tag{18.3}$$

$$W = \oint_S (t_{\ell k}v_k + m_{\ell k}\nu_k)da_\ell + \int_V \rho(f_k v_k + \ell_k\nu_k)dv \tag{18.4}$$

$$Q = \oint_S q_k da_k + \int_V \rho h dv \tag{18.5}$$

The physical meaning of some of the terms occurring in these equations is known to us in the classical continuum. For example, the first term in the integrand of (18.2) is the kinetic energy of the macromotion. The second term is, however, new and it is the kinetic energy of the microrotation. In (18.3), ε is the internal energy density per unit mass. In (18.4), the surface integral is the work of surface tractions and surface couples per unit time, while the volume integral is the work of the body force and body couple, per unit time.

Finally, in (18.5) the surface integral gives the heat input, and the volume integral the heat source.

The equation of local energy balance is obtained by postulating that (18.1) is valid for any arbitrary volume contained in the body. To this end we first carry out the indicated differentiation with respect to time for K and E. Thus

$$\dot{K} = \int_V \rho(a_k v_k + \dot{\sigma}_k \nu_k)dv \tag{18.6}$$

$$\dot{E} = \int_V \rho\dot{\varepsilon}dv \tag{18.7}$$

where, in anticipation of the local laws, we employed the equations of conservations of local mass and inertia $(17.5)_2$, (17.7), namely,

$$\dot{\overline{\rho dv}} = 0 \quad , \quad \frac{Dj_{k\ell}}{Dt} = 0 \tag{18.8}$$

Next we convert the surface integrals of (18.4) and (18.5) into volume integrals by use of the Green-Gauss theorem. Hence

$$W = \int_V (t_{\ell k}v_{k,\ell} + m_{\ell k}\nu_{k,\ell})dv + \int_V [(t_{\ell k,\ell} + \rho f_k)v_k + (m_{\ell k,\ell} + \rho \ell_k)\nu_k]dv$$

$$Q = \int_V (q_{k,k} + \rho h)dv$$

Upon carrying (18.6), (18.7), and the above equations into (18.1), and using the equations of local balance of momenta (17.17) and (17.18), we obtain

$$\int_V (\rho\dot{\varepsilon} - t_{\ell k}v_{k,\ell} + \varepsilon_{kmn}t_{mn}\nu_k - m_{\ell k}\nu_{k,\ell} - q_{k,k} - \rho h)dv = 0$$

This is assumed to be valid for every part of the body. Thus we must have

$$\rho\dot{\varepsilon} = t_{\ell k} v_{k,\ell} - \varepsilon_{kmn} t_{mn} \nu_k + m_{\ell k} \nu_{k,\ell} + q_{k,k} + \rho h \tag{18.9}$$

This is the differential equation of the <u>local balance of energy</u> of a micropolar body[1]. In expanded form it reads

$$
\begin{aligned}
\rho\dot{\varepsilon} = {} & t_{xx} \frac{\partial v_x}{\partial x} + t_{yx} \frac{\partial v_x}{\partial y} + t_{zx} \frac{\partial v_x}{\partial z} \\
& + t_{xy} \frac{\partial v_y}{\partial x} + t_{yy} \frac{\partial v_y}{\partial y} + t_{zy} \frac{\partial v_y}{\partial z} \\
& + t_{xz} \frac{\partial v_z}{\partial x} + t_{yz} \frac{\partial v_z}{\partial y} + t_{zz} \frac{\partial v_z}{\partial z} \\
& - (t_{yz} - t_{zy}) \nu_x - (t_{zx} - t_{xz}) \nu_y - (t_{xy} - t_{yx}) \nu_z \\
& + m_{xx} \frac{\partial \nu_x}{\partial x} + m_{yx} \frac{\partial \nu_x}{\partial y} + m_{zx} \frac{\partial \nu_x}{\partial z} \\
& + m_{xy} \frac{\partial \nu_y}{\partial x} + m_{yy} \frac{\partial \nu_y}{\partial y} + m_{zy} \frac{\partial \nu_y}{\partial z} \\
& + m_{xz} \frac{\partial \nu_z}{\partial x} + m_{yz} \frac{\partial \nu_z}{\partial y} + m_{zz} \frac{\partial \nu_z}{\partial z} \\
& + \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} + \rho h
\end{aligned}
\tag{18.10}
$$

We also note that

$$\dot{\varepsilon} = \frac{\partial \varepsilon}{\partial t} + \frac{\partial \varepsilon}{\partial x} v_x + \frac{\partial \varepsilon}{\partial y} v_y + \frac{\partial \varepsilon}{\partial z} v_z \tag{18.11}$$

---

[1] Again the energy balance equation (18.9) is exact in this form and is valid for the nonlinear theory.

19. PRINCIPLE OF ENTROPY

For certain classes of physical phenomena within a range of expected changes, a material body is characterized by certain constitutive equations. These equations define an ideal material approximating the real material under consideration. For any thermomechanical change, the constitutive equations assumed must not violate the second law of thermodynamics. In continuum mechanics, this law may be stated as follows:

V. Principle of Entropy (Clausius-Duhem Inequality). The time rate of change of the total entropy H is never less than the entropy influx through the surface $\mathcal{S}$ of the body and the volume entropy supply B in the body. This is postulated to be true for all parts of the body and for all independent processes (Eringen [1966c]).

Accordingly we write

$$\Gamma \equiv \frac{dH}{dt} - B - \oint_{\mathcal{S}} \underset{\sim}{S} \cdot \underset{\sim}{da} \geq 0 \tag{19.1}$$

where Γ so defined is the total entropy production. For simple thermomechanical processes we have

$$H \equiv \int_{\mathcal{V}} \rho\eta dv \tag{19.2}$$

$$B \equiv \int_{\mathcal{V}} \frac{h}{\theta} dv \tag{19.3}$$

$$\underset{\sim}{S} \equiv \frac{\underset{\sim}{q}}{\theta} \tag{19.4}$$

where η, h, $\underset{\sim}{q}$ and θ are, respectively, the entropy density, heat source, heat vector, and the absolute temperature. Substituting (19.2) to (19.4) into (19.1) gives

$$\Gamma \equiv \frac{d}{dt}\int_V \rho\eta dv - \int_V \frac{h}{\theta} dv - \oint_S \frac{q_k}{\theta} da_k \geq 0 \tag{19.5}$$

Using the Green-Gauss theorem to convert the surface integral to a volume integral and carrying out the differentiation with respect to time, we get

$$\Gamma \equiv \int_V [\rho\dot{\eta} - (\frac{q_k}{\theta})_{,k} - \frac{h}{\theta}]dv \geq 0 \tag{19.6}$$

Since this is to be valid for all parts of the body, we must have

$$\rho\dot{\eta} - (\frac{q_k}{\theta})_{,k} - \frac{h}{\theta} \geq 0 \tag{19.7}$$

This is the well-known Clausius-Duhem inequality of classical continuum mechanics. In micropolar bodies, this is considered to be unchanged.

Upon substituting h solved from (18.9), we can rearrange (19.7) into the following form

$$\begin{aligned} \rho\gamma \equiv \rho(\dot{\eta} - \frac{\dot{\varepsilon}}{\theta}) + \frac{1}{\theta} t_{k\ell} v_{\ell,k} - \frac{1}{\theta} \varepsilon_{kmn} t_{mn} \nu_k \\ + \frac{1}{\theta} m_{\ell k} \nu_{k,\ell} + \frac{1}{\theta^2} q_k \theta_{,k} \geq 0 \end{aligned} \tag{19.8}$$

Still another form, convenient for some cases, is found by introducing the <u>Helmholtz free energy</u>

$$\psi = \varepsilon - \theta\eta \tag{19.9}$$

Hence

$$\begin{aligned} \rho\gamma \equiv -\frac{\rho}{\theta}(\dot{\psi} + \eta\dot{\theta}) + \frac{1}{\theta} t_{k\ell} v_{\ell,k} - \frac{1}{\theta} \varepsilon_{kmn} t_{mn} \nu_k \\ + \frac{1}{\theta} m_{\ell k} \nu_{k,\ell} + \frac{1}{\theta^2} q_k \theta_{,k} \geq 0 \end{aligned} \tag{19.10}$$

The Clausius-Duhem inequality (19.8) or (19.10) is postulated to be valid for all independent thermomechanical changes. This implies that we must know the independent variables which affect χ, η, $t_{k\ell}$, $m_{k\ell}$, and q_k at the outset. This in turn requires writing constitutive equations for these variables. An example of this is to be found in the following article.

We also note that the entropy inequality (19.5) can be shown to lead to an inequality restricting the normal component of $\mathbf{q}/\theta$ on the surface of the body (Eringen [1966c]), i.e.,

$$[\frac{\mathbf{q}}{\theta}] \cdot \mathbf{n} \geq 0 \qquad \text{on } S \tag{19.11}$$

where a boldface bracket indicates the difference on the quantity enclosed calculated from two different sides of S, on S.

20. THEORY OF MICROPOLAR ELASTICITY

A micropolar elastic solid is distinguished from an elastic solid by the fact that it can support body and surface couples. These solids can undergo local deformations and microrotations. Such materials may be imagined as bodies which are made of rigid short cylinders or dumbbell type molecules.

From a continuum mechanical point of view, micropolar elastic solids may be characterized by a set of constitutive equations which define the elastic properties of such materials. A linear theory as a special case of the nonlinear theory of microelastic solids was first constructed by Eringen and Suhubi [1964a,b]. Later, Eringen [1965], [1966] reorganized and extended this theory. Here we give a self-contained account of this theory.

In linear micropolar elasticity, the strain measures are (cf. equations (4.17) and (4.18))

(20.1) $$\varepsilon_{k\ell} = e_{k\ell} + \varepsilon_{k\ell m}(r_m - \phi_m) = u_{\ell,k} + \varepsilon_{\ell km}\phi_m$$

(20.2) $$\gamma_{k\ell m} = \varepsilon_{k\ell n}\phi_{n,m}$$

Since only the nine components $\phi_{k,\ell}$ of $\gamma_{k\ell m}$ are independent and non-vanishing (cf. equation (4.37)), we may instead of $\gamma_{k\ell m}$ use the axial tensor $\phi_{k,\ell}$ for simplicity. Upon arbitrary rotations and reflections of the spatial coordinates represented by $Q_{k\ell}$, i.e.,

(20.3) $$x'_k = Q_{k\ell}x_\ell$$

(20.4) $$Q_{k\ell}Q_{m\ell} = Q_{\ell k}Q_{m\ell} = \delta_{km} \quad , \quad \det Q_{k\ell} = \pm 1$$

and the tensors $\varepsilon_{k\ell}$ and $\phi_{k,\ell}$ transform according to

$$\varepsilon'_{k\ell} = Q_{km}\varepsilon_{mn}Q_{\ell n}$$

(20.5)

$$\phi'_{k,\ell} = \pm Q_{km}\phi_{m,n}Q_{\ell n}$$

where in the last equation the plus sign is for det $Q_{k\ell} = +1$ and the minus sign for det $Q_{k\ell} = -1$. This is because ϕ_k is an axial vector. Equations (20.5) express the fact that both $\varepsilon_{k\ell}$ and $\phi_{k,\ell}$ are <u>objective</u> tensors and are appropriate for use as independent constitutive variables. To this list of variables we also include the temperature θ so that the material properties of these materials may depend on the temperature as well. The constitutive dependent variables are

$$t_{k\ell},\ m_{k\ell},\ q_k,\ \psi \text{ and } \eta$$

We now propose a set of constitutive equations of the form

$$t_{k\ell} = F_{k\ell}(\varepsilon_{rs},\ \phi_{r,s},\ \theta)$$

$$m_{k\ell} = M_{k\ell}(\varepsilon_{rs},\ \phi_{r,s},\ \theta)$$

(20.6)

$$q_k = G_k(\varepsilon_{rs},\ \phi_{r,s},\ \theta)$$

$$\psi = \Psi(\varepsilon_{rs},\ \phi_{r,s},\ \theta)$$

$$\eta = N(\varepsilon_{rs},\ \phi_{r,s},\ \theta)$$

The above equations are legitimate for linear homogeneous materials whether isotropic or not. For nonlinear isotropic materials they are acceptable in form. However, since we are employing the infinitesimal strain measures, a nonlinear constitutive theory in terms of linear strain measures would be inconsistent. For the nonlinear theory, the reader is referred to Eringen and Suhubi [1964a & b].

The constitutive equation (20.6) must be consistent with the second law of thermodynamics as expressed by (19.10). Thus, upon substituting (20.6) into (19.10), we have

$$
\begin{aligned}
&-\frac{\rho}{\theta}\left(\frac{\partial\Psi}{\partial\varepsilon_{k\ell}}\dot{\varepsilon}_{k\ell} + \frac{\partial\Psi}{\partial\phi_{k,\ell}}\dot{\phi}_{k,\ell} + \frac{\partial\Psi}{\partial\theta}\dot{\theta} + \eta\dot{\theta}\right) + \frac{1}{\theta}t_{k\ell}\dot{\varepsilon}_{k\ell} \\
&\qquad + \frac{1}{\theta}m_{k\ell}\dot{\phi}_{\ell,k} + \frac{1}{\theta^2}q_k\theta_{,k} \geq 0
\end{aligned} \tag{20.7}
$$

Consistent with the linear theory we write

$$
\begin{aligned}
&\frac{D}{Dt}(\phi_{k,\ell}) = \dot{\phi}_{k,\ell} \\
&\dot{\varepsilon}_{k\ell} = v_{k,\ell} - \varepsilon_{k\ell m}\nu_m
\end{aligned} \tag{20.8}
$$

The inequality (20.7) is postulated to be valid for all independent processes. Here $\dot{\varepsilon}_{k\ell}$, $\dot{\phi}_{k,\ell}$, $\dot{\theta}$ and $\theta_{,k}$ can be varied independently. Since this inequality is linear in all these variables, we must set the coefficients of these variables equal to zero. Hence

$$
\begin{aligned}
t_{k\ell} &= \rho\frac{\partial\Psi}{\partial\varepsilon_{k\ell}} \\
m_{k\ell} &= \rho\frac{\partial\Psi}{\partial\phi_{\ell,k}} \\
q_k &= 0 \\
\eta &= -\frac{\partial\Psi}{\partial\theta}
\end{aligned} \tag{20.9}
$$

We therefore see that for a micropolar elastic solid stress, the couple stress and entropy density are derivable from a potential and the heat vector vanishes. Since we did not consider the temperature gradient, we have no heat conduction. Nevertheless, the free energy Ψ and consequently the material moduli will depend on the temperature θ. Since all terms in (20.7) vanish, we have the entropy production density γ also vanishing. Thus, the micropolar elastic solid is in thermal equilibrium.

Here we are concerned with the linear theory. We therefore consider a polynomial for Φ which is second degree in the strain measures $\varepsilon_{k\ell}$ and $\phi_{k,\ell}$, i.e.,

$$\rho\Psi = A_0 + A_{k\ell}\varepsilon_{k\ell} + \frac{1}{2} A_{k\ell mn}\varepsilon_{mn} + B_{k\ell}\phi_{k,\ell} + \frac{1}{2} B_{k\ell mn}\phi_{k,\ell}\phi_{m,n} + C_{k\ell mn}\varepsilon_{k\ell}\phi_{m,n} \tag{20.10}$$

where A_0, $A_{k\ell}$, $A_{k\ell mn}$, $B_{k\ell}$, ... are functions of θ only. Since ϕ_k is an axial vector, upon a reflection of the spatial axes the fourth and the last terms will change sign while the other terms do not. For the function Ψ to be invariant $B_{k\ell} = 0$, $C_{k\ell mn} = 0$. We further note the following symmetry conditions which are clear from various summations in (20.10)

$$A_{k\ell mn} = A_{mnk\ell} \quad , \quad B_{k\ell mn} = B_{mnk\ell} \tag{20.11}$$

which shows that for the most general micropolar anisotropic elastic solid, the number of $A_{k\ell mn}$ and $B_{mnk\ell}$ is 45 each. In addition, we have nine $A_{k\ell}$ which give rise to an initial stress in the undeformed state of the body.

Upon substituting (20.10) into $(20.9)_1$ and $(20.9)_2$, we obtain

$$t_{k\ell} = A_{k\ell} + A_{\ell kmn}\varepsilon_{mn} \tag{20.12}$$

$$m_{k\ell} = B_{\ell kmn}\phi_{m,n} \tag{20.13}$$

These are the linear forms of the stress and couple stress constitutive equations for the anisotropic micropolar elastic solids. When the initial stress is zero, then we must also have $A_{k\ell} = 0$. Thus, for the micropolar solid free of initial stress and couple stress, we have

$$t_{k\ell} = A_{k\ell mn}\varepsilon_{mn} \tag{20.14}$$

$$m_{k\ell} = B_{\ell kmn}\phi_{m,n} \tag{20.15}$$

Various material symmetry conditions place further restrictions on the constitutive coefficients $A_{\ell kmn}$ and $B_{\ell kmn}$. These restrictions are found in the same manner as in classical elasticity. Here we obtain the case of <u>isotropic</u> solids. If the body is isotropic with respect to both the stress and couple stress, we call the solid <u>microisotropic</u>. In this case, the constitutive coefficients must be isotropic tensors. For the second and fourth-order isotropic tensors, we have the most general forms

$$\begin{gathered} A_{k\ell} = A\delta_{k\ell}, \; A_{k\ell mn} = A_1\delta_{k\ell}\delta_{mn} + A_2\delta_{km}\delta_{\ell n} + A_3\delta_{kn}\delta_{\ell m} \\ B_{k\ell mn} = B_1\delta_{k\ell}\delta_{mn} + B_2\delta_{km}\delta_{\ell n} + B_3\delta_{kn}\delta_{\ell m} \end{gathered} \tag{20.16}$$

where A, A_1, A_2, A_3, B_1, B_2, and B_3 are functions of θ only. In this case then, (20.12) and (20.13) take the special forms

$$t_{k\ell} = A\delta_{k\ell} + A_1\varepsilon_{rr}\delta_{k\ell} + A_2\varepsilon_{k\ell} + A_3\varepsilon_{\ell k} \tag{20.17}$$

$$m_{k\ell} = B\delta_{k\ell} + B_1\phi_{r,r}\delta_{k\ell} + B_2\phi_{\ell,k} + B_3\phi_{k,\ell} \tag{20.18}$$

For vanishing initial stress $A = 0$. Introducing

$$A_1 \equiv \lambda\ , \quad A_2 \equiv \mu + \kappa\ , \quad A_3 \equiv \mu$$

(20.19)

$$B_1 \equiv \alpha\ , \quad B_2 \equiv \gamma \quad , \quad B_3 \equiv \beta$$

the above equations can be written as

(20.20)
$$t_{k\ell} = \lambda\varepsilon_{rr}\delta_{k\ell} + (\mu + \kappa)\varepsilon_{k\ell} + \mu\varepsilon_{\ell k}$$

(20.21)
$$m_{k\ell} = \alpha\phi_{r,r}\delta_{k\ell} + \beta\phi_{k,\ell} + \gamma\phi_{\ell,k}$$

For the free energy in this case we find

(20.22)
$$\rho\Psi = \frac{1}{2}\,[\lambda\varepsilon_{kk}\varepsilon_{\ell\ell} + (\mu + \kappa)\varepsilon_{k\ell}\varepsilon_{k\ell} + \mu\varepsilon_{k\ell}\varepsilon_{\ell k}]$$
$$+\frac{1}{2}\,(\alpha\phi_{k,k}\phi_{\ell,\ell} + \beta\phi_{k,\ell}\phi_{\ell,k} + \gamma\phi_{k,\ell}\phi_{k,\ell})$$

An alternative form to (20.20) to (20.22) is

(20.23)
$$t_{k\ell} = \lambda e_{rr}\delta_{k\ell} + (2\mu + \kappa)e_{k\ell} + \kappa\varepsilon_{k\ell m}(r_m - \phi_m)$$

(20.24)
$$m_{k\ell} = \alpha\phi_{r,r}\delta_{k\ell} + \beta\phi_{k,\ell} + \gamma\phi_{\ell,k}$$

(20.25)
$$\rho\Psi = \frac{1}{2}\,[\lambda e_{kk}e_{\ell\ell} + (2\mu + \kappa)e_{k\ell}e_{k\ell}] + \kappa(r_k - \phi_k)(r_k - \phi_k)$$
$$+ \frac{1}{2}\,(\alpha\phi_{k,k}\phi_{\ell,\ell} + \beta\phi_{k,\ell}\phi_{\ell,k} + \gamma\phi_{k,\ell}\phi_{k,\ell})$$

We note the difference between isotropic micropolar elasticity and classical elasticity by the presence of four extra elastic moduli, namely, κ, α, β and γ. When these are set equal to zero, the above equations (20.23) to (20.25) revert to Hooke's law of the linear isotropic elastic solid.

21. RESTRICTIONS ON MICROPOLAR ELASTIC MODULI

The stability of materials requires that the stored elastic energy be nonnegative. This condition is also essential for the uniqueness of the solutions. This requirement places certain restrictions on the micropolar elastic moduli. The following theorem, Eringen [1966a], provides these conditions for ψ independent of θ.

Theorem. The necessary and sufficient conditions for the internal energy to be nonnegative are

$$\begin{aligned} &0 \leq 3\lambda + 2\mu + \kappa\,, \quad 0 \leq \mu\,, \quad 0 \leq \kappa \\ &0 \leq 3\alpha + 2\gamma\,, \quad -\gamma \leq \beta \leq \gamma, \quad 0 \leq \gamma \end{aligned} \tag{21.1}$$

The sufficiency of (21.1) is proven by observing that when these inequalities hold, each one of the following energies constituting the internal energy density is nonnegative

$$\varepsilon = \varepsilon_E + \varepsilon_R + \varepsilon_M \tag{21.2}$$

where

$$\begin{aligned} \rho\varepsilon_E &\equiv \frac{1}{2}\,[\lambda e_{kk}e_{\ell\ell} + (2\mu + \kappa)e_{k\ell}e_{\ell k}] \\ \rho\varepsilon_R &\equiv \kappa(r_k - \phi_k)(r_k - \phi_k) \\ \rho\varepsilon_M &= \frac{1}{2}\,(\alpha\phi_{k,k}\phi_{\ell,\ell} + \beta\phi_{k,\ell}\phi_{\ell,k} + \gamma\phi_{\ell,k}\phi_{\ell,k}) \end{aligned} \tag{21.3}$$

The fact that $\rho\varepsilon_E$ is nonnegative under the conditions $(21.1)_1$ and $(21.1)_2$ is well known for the classical elasticity. It is simple to observe that $\rho\varepsilon_R$ is nonnegative for $r_k \neq \phi_k$ whenever $\kappa \geq 0$. To see the same for $\rho\varepsilon_M$, we write this expression as

$$2\rho\varepsilon_M = \frac{1}{3}(3\alpha + \beta + \gamma)\phi_{k,k}\phi_{\ell,\ell} + (\gamma - \beta)\phi_{[k,\ell]}\phi_{[k,\ell]}$$

(21.4)

$$+ (\gamma + \beta)[\phi_{(k,\ell)} - \frac{1}{3}\phi_{r,r}\phi_{k\ell}][\phi_{(k,\ell)} - \frac{1}{3}\phi_{s,s}\delta_{k\ell}]$$

where

$$\phi_{(k,\ell)} \equiv \frac{1}{2}(\phi_{k,\ell} + \phi_{\ell,k})$$

(21.5)

$$\phi_{[k,\ell]} \equiv \frac{1}{2}(\phi_{k,\ell} - \phi_{\ell,k})$$

From (21.4) it is clear that when

(21.6) $$3\alpha + \beta + \gamma \geq 0 \quad , \quad \gamma - \beta \geq 0 \quad , \quad \gamma + \beta \geq 0$$

we have $\rho\varepsilon_M \geq 0$ so that (21.1) are sufficient for $\rho\varepsilon_M \geq 0$.

Conditions (21.1) are also necessary for the nonnegativeness of $\rho\varepsilon$. To prove this, we recall that $e_{k\ell}$, $r_k - \phi_k$ and $\phi_{k,\ell}$ can be varied independently of each other. Since the above three energies are uncoupled with respect to these variables, each one of these energies must be nonnegative independent of each other. The fact that $(21.1)_1$ and $(21.1)_2$ are necessary for $\rho\varepsilon_E \geq 0$ is known to us from classical elasticity. Excluding the case of $r_k = \phi_k$ (<u>indeterminate couple stress theory</u>), we see that $\rho\varepsilon_R$ is nonnegative if and only if $\kappa \geq 0$. For the case of $r_k = \phi_k$, by replacing $2\mu + \kappa$ by a new modulus 2μ we shall have κ disappear from the constitutive equations. Thus, it remains to prove the necessity of (21.1) for the nonnegativeness of $\rho\varepsilon_M$. To this we write it as a quadratic form in a nine-dimensional space, i.e.,

(21.7) $$\rho\varepsilon_M = a_{ij}y_iy_j \quad , \quad a_{ij} = a_{ji} \quad , \quad (i,j = 1,2,\ldots,9)$$

where

$$y_1 \equiv \phi_{1,1} \;, \quad y_2 \equiv \phi_{2,2} \;, \quad y_3 \equiv \phi_{3,3}$$

$$y_4 \equiv \phi_{1,2} \;, \quad y_5 \equiv \phi_{2,1} \;, \quad y_6 \equiv \phi_{2,3}$$

(21.8) $$y_7 \equiv \phi_{3,2} \;, \quad y_8 \equiv \phi_{3,1} \;, \quad y_9 \equiv \phi_{1,3}$$

$$a_{11} = a_{22} = a_{33} = \alpha + \beta + \gamma, \quad a_{44} = a_{55} = a_{66} = a_{77} = a_{88} = a_{99} = \gamma$$

$$a_{12} = a_{13} = a_{23} = \alpha \quad , \quad \alpha_{45} = a_{67} = a_{89} = \beta$$

all other $a_{ij} = 0$

The characteristic values a_i of a_{ij} are obtained by solving the equation

(21.9) $$\det\,(a_{ij} - a\delta_{ij}) = 0$$

The nine roots a_i of this equation are

$$a_1 = a_2 = a_3 = \gamma - \beta \quad , \quad a_4 = a_5 = a_6 = a_7 = a_8 = \gamma + \beta$$

$$a_9 = 3\alpha + \beta + \gamma$$

In order for $\rho\varepsilon_M \geq 0$ to be satisfied for all y_i, it is necessary (and sufficient that) (21.7) be an ellipsoid in nine-dimensional space, i.e.,

$$\gamma - \beta \geq 0 \quad , \quad \gamma + \beta \geq 0 \quad , \quad 3\alpha + \beta + \gamma \geq 0$$

This set of conditions is the same as the last three conditions of (21.1). Hence the proof of the theorem.

The nonnegative character of the internal energy density has important implications in regard to uniqueness theorems in both static and dynamic micropolar elasticity. For these and other important results see Eringen [1966a,b].

22. FIELD EQUATIONS, BOUNDARY AND INITIAL CONDITIONS

The field equations of linear micropolar elasticity are obtained by substituting (20.23) and (20.24) into (17.17) and (17.18). Hence

$$(22.1)\qquad (\lambda + \mu)u_{\ell,\ell k} + (\mu + \kappa)u_{k,\ell\ell} + \kappa\varepsilon_{k\ell m}\phi_{m,\ell} + \rho(f_k - \ddot{u}_k) = 0$$

$$(22.2)\qquad (\alpha + \beta)\phi_{\ell,\ell k} + \gamma\phi_{k,\ell\ell} + \kappa\varepsilon_{k\ell m}u_{m,\ell} - 2\kappa\phi_k + \rho(\ell_k - j\ddot{\phi}_k) = 0$$

where we have taken $j_{k\ell} = j\delta_{k\ell}$ for the microisotropic solid. In the linear theory, ρ and j are considered constants and the accelerations $\ddot{u}_k$ and $\ddot{\phi}_k$ are calculated by their approximate expressions

$$(22.3)\qquad \ddot{u}_k \simeq \frac{\partial^2 u_k}{\partial t^2} \quad , \quad \ddot{\phi}_k \simeq \frac{\partial^2 \phi_k}{\partial t^2}$$

The vectorial forms of these equations are found to be convenient for the treatment of problems in curvilinear coordinates. These are readily obtained from the above equations by simply multiplying them by $\underset{\sim}{i}_k$ and observing that

$$(22.4)\qquad u_{\ell,\ell k}\underset{\sim}{i}_k = \underset{\sim}{\nabla}\underset{\sim}{\nabla} \cdot \underset{\sim}{u} \quad , \quad \varepsilon_{k\ell m}\phi_{m,\ell}\underset{\sim}{i}_k = \nabla \times \underset{\sim}{\phi}$$

$$u_{k,\ell\ell}\underset{\sim}{i}_k = \underset{\sim}{\nabla}\underset{\sim}{\nabla} \cdot \underset{\sim}{u} - \underset{\sim}{\nabla} \times \nabla \times \underset{\sim}{u}$$

where $\underset{\sim}{\nabla}$ is the gradient operator so that

$$\underset{\sim}{\nabla}\phi \equiv \text{grad}\ \phi \quad , \quad \underset{\sim}{\nabla} \cdot \underset{\sim}{u} \equiv \text{div}\ \underset{\sim}{u} \quad , \quad \underset{\sim}{\nabla} \times \underset{\sim}{u} \equiv \text{curl}\ \underset{\sim}{u}$$

Hence

$$(22.5)\qquad (\lambda + 2\mu + \kappa)\underset{\sim}{\nabla}\underset{\sim}{\nabla} \cdot u - (\mu + \kappa)\underset{\sim}{\nabla} \times \underset{\sim}{\nabla} \times \underset{\sim}{u} + \kappa\underset{\sim}{\nabla} \times \underset{\sim}{\phi} + \rho(\underset{\sim}{f} - \ddot{\underset{\sim}{u}}) = 0$$

$$(22.6)\qquad (\alpha + \beta + \gamma)\underset{\sim}{\nabla}\underset{\sim}{\nabla} \cdot \underset{\sim}{\phi} - \gamma\underset{\sim}{\nabla} \times \underset{\sim}{\nabla} \times \underset{\sim}{\phi} + \kappa\underset{\sim}{\nabla} \times \underset{\sim}{u} - 2\kappa\underset{\sim}{\phi} + \rho(\underset{\sim}{\ell} - j\ddot{\underset{\sim}{\phi}}) = 0$$

For an initial value problem, the <u>initial conditions</u> have the form

$$(22.7) \qquad \underset{\sim}{u}(\underset{\sim}{x},0) = \underset{\sim}{u}_0(\underset{\sim}{x}) \quad , \quad \dot{\underset{\sim}{u}}(\underset{\sim}{x},0) = \underset{\sim}{v}_0(\underset{\sim}{x})$$

in V

$$(22.8) \qquad \underset{\sim}{\phi}(\underset{\sim}{x},0) = \underset{\sim}{\phi}_0(\underset{\sim}{x}) \quad , \quad \dot{\underset{\sim}{\phi}}(\underset{\sim}{x},0) = \underset{\sim}{\nu}_0(\underset{\sim}{x})$$

where $\underset{\sim}{u}_0$, $\underset{\sim}{v}_0$, $\underset{\sim}{\phi}_0$, and $\underset{\sim}{\nu}_0$ are prescribed in V at time $t = 0$.

Many different types of boundary conditions are suggested in applications. For example, we may prescribe

$$(22.9) \qquad \underset{\sim}{u}(\underset{\sim}{x}',t) = \underset{\sim}{u}'$$

$\underset{\sim}{x}'$ on S

$$(22.10) \qquad \underset{\sim}{\phi}(\underset{\sim}{x}',t) = \underset{\sim}{\phi}'$$

on the boundary surface S of the body. An equally permissible set of boundary conditions requires the prescription of the tractions and couples, i.e.,

$$(22.11) \qquad t_{\ell k} n_\ell = t_{(\underset{\sim}{n})k}$$

on S

$$(22.12) \qquad m_{\ell k} n_\ell = m_{(\underset{\sim}{n})k}$$

where $t_{\ell k}$ and $m_{\ell k}$ are the stress and the couple stress tensors given by (20.23) and (20.24) and $t_{(\underset{\sim}{n})k}$ and $m_{(\underset{\sim}{n})k}$ are the prescribed tractions and couples on S whose exterior normal is $\underset{\sim}{n}$.

In some other problems, a mixture of the above two types of conditions occurs, e.g., on some part S_d of S one may have (22.9) and (22.10) and on the remainder $S_\ell \equiv S - S_d$ the conditions (22.11) and (22.12). Still other types of mixed conditions involving some components of one set and the remaining components of the other set are possible. All admissible sets of boundary conditions allowing unique solutions must satisfy (Eringen [1966a])

$$\bar{t}_{(\underset{\sim}{n})k}\dot{\bar{u}}_k + \bar{m}_{(\underset{\sim}{n})k}\dot{\bar{\phi}}_k = 0 \quad \text{on } S \quad , \qquad t \geq 0 \tag{22.13}$$

where $\bar{\underset{\sim}{u}}$, $\bar{\underset{\sim}{\phi}}$, $\bar{\underset{\sim}{t}}_{(\underset{\sim}{n})}$, and $\bar{\underset{\sim}{m}}_{(\underset{\sim}{n})}$ are, respectively, the difference of $\underset{\sim}{u}$, $\underset{\sim}{\phi}$, $\underset{\sim}{t}_{(\underset{\sim}{n})}$ and $\underset{\sim}{m}_{(\underset{\sim}{n})}$ from their respective values on S.

The field equations (22.1) and (22.2) are valid only for micropolar isotropic solids. Note that for vanishing κ, α, β, $\underset{\sim}{\ell}$, and j, equation (22.2) reduces to $\underset{\sim}{0} = \underset{\sim}{0}$, and (22.1) gives the celebrated equations of Navier of classical elasticity.

For the anisotropic micropolar elastic solid, the field equations replacing (22.1) and (22.2) are obtained by substituting (20.14) and (20.15) into (17.17) and (17.18)

$$A_{\ell kmn}(u_{n,m\ell} + \varepsilon_{nmr}\phi_{r,\ell}) + \rho(f_k - \ddot{u}_k) = 0 \tag{22.14}$$

$$B_{k\ell mn}\phi_{m,n\ell} + \varepsilon_{kmn}A_{mnpq}(u_{q,p} + \varepsilon_{qpr}\phi_r) + \rho(\ell_k - j_{\ell k}\ddot{\phi}_\ell) = 0 \tag{22.15}$$

For the expressions of $\varepsilon_{\ell k}$ and σ_k we used (20.1) and (15.18).

Finally, we record below the expanded forms of the field equations (22.1) and (22.2) in rectangular coordinates.

$$\begin{aligned}
&(\lambda + \mu)\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + (\mu + \kappa)\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right) \\
&\qquad + \kappa\left(\frac{\partial \phi_z}{\partial y} - \frac{\partial \phi_y}{\partial z}\right) + \rho\left(f_x - \frac{\partial^2 u_x}{\partial t^2}\right) = 0 \\
&(\lambda + \mu)\frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + (\mu + \kappa)\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2}\right) \\
&\qquad + \kappa\left(\frac{\partial \phi_x}{\partial z} - \frac{\partial \phi_z}{\partial x}\right) + \rho\left(f_y - \frac{\partial^2 u_y}{\partial t^2}\right) = 0 \\
&(\lambda + \mu)\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + (\mu + \kappa)\left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2}\right) \\
&\qquad + \kappa\left(\frac{\partial \phi_y}{\partial x} - \frac{\partial \phi_x}{\partial y}\right) + \rho\left(f_z - \frac{\partial^2 u_z}{\partial t^2}\right) = 0
\end{aligned} \tag{22.16}$$

$$(\alpha + \beta) \frac{\partial}{\partial x} \left(\frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y} + \frac{\partial \phi_z}{\partial z}\right) + \gamma\left(\frac{\partial^2 \phi_x}{\partial x^2} + \frac{\partial^2 \phi_x}{\partial y^2} + \frac{\partial^2 \phi_x}{\partial z^2}\right) + \kappa\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)$$

$$- 2\kappa\phi_x + \rho\left(\ell_x - j \frac{\partial^2 \phi_x}{\partial t^2}\right) = 0$$

$$(\alpha + \beta) \frac{\partial}{\partial x} \left(\frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y} + \frac{\partial \phi_z}{\partial z}\right) + \gamma\left(\frac{\partial^2 \phi_y}{\partial x^2} + \frac{\partial^2 \phi_y}{\partial y^2} + \frac{\partial^2 \phi_y}{\partial z^2}\right) + \kappa\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)$$

$$- 2\kappa\phi_y + \rho\left(\ell_y - j \frac{\partial^2 \phi_y}{\partial t^2}\right) = 0 \qquad (22.17)$$

$$(\alpha + \beta) \frac{\partial}{\partial z} \left(\frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y} + \frac{\partial \phi_z}{\partial z}\right) + \gamma\left(\frac{\partial^2 \phi_z}{\partial x^2} + \frac{\partial^2 \phi_z}{\partial y^2} + \frac{\partial^2 \phi_z}{\partial z^2}\right) + \kappa\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)$$

$$- 2\kappa\phi_z + \rho\left(\ell_z - j \frac{\partial^2 \phi_z}{\partial t^2}\right) = 0$$

23. INDETERMINATE COUPLE STRESS THEORY

In Art. 22 we indicated that classical elasticity is a special case of the micropolar theory when $\kappa = \alpha = \beta = \gamma = j = 0$ and $\underset{\sim}{\ell} = \underset{\sim}{0}$. The same is also true for motions in which the microrotation vanishes. There exist other classes of constrained motions which have attracted the attention of research workers. The most popular among them is the indetermine couple stress theory which is contained in the work of the Cosserats [1909]. Recently, Truesdell and Toupin [1960], Grioli [1960], Aero and Kuvshinskii [1960], Mindlin and Tiersten [1962], Toupin [1962], and Eringen [1962] independently presented new derivations and supplied various missing parts of the theory. This theory can be obtained as a special case of the micropolar theory if the constraints

$$\phi_k = r_k = \frac{1}{2} \varepsilon_{k\ell m} u_{m,\ell} \tag{23.1}$$

are imposed. In this case, the stress constitutive equations (20.23) reduce to

$$t_{(k\ell)} = \lambda e_{rr} \delta_{k\ell} + (2\mu + \kappa) e_{k\ell} \tag{23.2}$$

where a parenthesis enclosing indices as usual indicates the symmetric part of the stress tensor. We also use a bracket to denote the antisymmetric part of tensors, e.g.,

$$a_{(k\ell)} \equiv \frac{1}{2} (a_{k\ell} + a_{\ell k}) \quad , \quad a_{[k\ell]} \equiv \frac{1}{2} (a_{k\ell} - a_{\ell k})$$

Thus, when (23.1) is valid, the antisymmetric part of the stress disappears from the constitutive equations. We can, however, remedy this situation by another artifice. The equations of moment of momentum (17.18) can be used to solve for the antisymmetric part of the stress tensor. Multiplying (17.18) by ε_{krs}, we solve for

$$t_{[k\ell]} = -\frac{1}{2}\varepsilon_{rk\ell}m_{nr,n} - \frac{1}{2}\rho\varepsilon_{rk\ell}(\ell_r - \dot{\sigma}_r) \tag{23.3}$$

where we used the identity

$$\varepsilon_{rk\ell}\varepsilon_{rmn} = \delta_{km}\delta_{\ell n} - \delta_{kn}\delta_{\ell m} \tag{23.4}$$

Upon carrying (23.1) into (20.24) we have

$$m_{k\ell} = \frac{\beta}{2}\varepsilon_{krs}u_{s,r\ell} + \frac{\gamma}{2}\varepsilon_{\ell rs}u_{s,rk} \tag{23.5}$$

Carrying (23.5) and the expression of

$$\dot{\sigma}_r = j\ddot{\phi}_r = \frac{1}{2}j\varepsilon_{rmn}\ddot{u}_{n,m} \tag{23.6}$$

into (23.3), we have

$$t_{[k\ell]} = \frac{\gamma}{2}\nabla^2 u_{[k,\ell]} - \frac{1}{2}\rho(\varepsilon_{rk\ell}\ell_r + j\ddot{u}_{[k,\ell]}) \tag{23.7}$$

where ∇^2 is the laplacian operator in rectangular coordinates

$$\nabla^2 u_\ell \equiv u_{\ell,kk} \tag{23.8}$$

If we not substitute (23.2) and (23.7) into

$$t_{k\ell} = t_{(k\ell)} + t_{[k\ell]} \tag{23.9}$$

we get the total stress tensor

$$t_{k\ell} = \lambda u_{r,r}\delta_{k\ell} + (\mu + \frac{\kappa}{2})(u_{k,\ell} + u_{\ell,k}) + \frac{\gamma}{2}\nabla^2 u_{[k,\ell]} - \frac{1}{2}\rho(\varepsilon_{rk\ell}\ell_r + j\ddot{u}_{[k,\ell]}) \tag{23.10}$$

The presence of the body couple ℓ_r and the acceleration $\ddot{u}_k$ in this equation is certainly disturbing since the constitutive equations, in general, should not contain such terms. When (23.9) is carried into the equation of balance of momentum (17.17), we find

$$(23.11) \qquad (\lambda + \mu + \frac{\kappa}{2} + \frac{\gamma}{4}\nabla^2)u_{k,\ell k} + (\mu + \frac{\kappa}{2} - \frac{\gamma}{4}\nabla^2)u_{\ell,kk}$$

$$+ \frac{\rho}{2}\varepsilon_{k\ell r}\ell_{r,\ell} + \rho f_\ell = \rho(1 - \frac{j}{4}\nabla^2)\ddot{u}_\ell + \frac{\rho j}{4}\ddot{u}_{k,\ell k}$$

By use of identities (22.4), we may also obtain the vector form of these equations:

$$(23.12) \qquad (\lambda + 2\mu + \kappa)\nabla\nabla \cdot \mathbf{u} - (\mu + \frac{\kappa}{2} - \frac{\gamma}{4}\nabla^2)\nabla \times \nabla \times \mathbf{u}$$

$$+ \rho(\mathbf{f} + \frac{1}{2}\nabla \times \boldsymbol{\ell}) - \rho(1 + \frac{j}{4}\nabla \times \nabla\times)\ddot{\mathbf{u}} = \mathbf{0}$$

where for the lapalcian operator we have used

$$(23.13) \qquad \nabla^2\mathbf{A} = \nabla\nabla \cdot \mathbf{A} - \nabla \times \nabla \times \mathbf{A}$$

Equation (23.12) takes the form (3.27) obtained by Mindlin and Tiersten [1962] in an entirely different way if we write μ for $\mu + \kappa/2$ and η for $\gamma/4$ and $j = 0$. Thus, these authors, as with others, have neglected the micropolar rotatory inertia. Equations (23.11) or (23.12) are the field equations of the theory known as the (indeterminate) couple stress theory. It is to be observed that in this theory the skew-symmetric part of stress and consequently the stress are dependent on the applied loads and inertia, and they are not determined solely as a result of the constitutive character of the medium. This violates the axiom objectivity since the applied loads and inertia terms involved are not objective quantities. A second relevant point is that while according to

(23.5) one can determine both the symmetric and the skew-symmetric parts of the couple stress, this in fact is not the case of the couple stress theory in which constitutive equations are derived from a free energy function, as is done in Art. 20. It is clear from $(20.9)_2$, for example, that all components of $\phi_{\ell,k}$ can no longer be used as independent variables. In fact, if one uses (23.1) in the argument of free energy Ψ, one finds that all nine components of $m_{k\ell}$ are not independent. Moreover, the skew-symmetric parts of the stress and couple stress remain indeterminate[1]. This is the reason for the use of the terminology "indeterminate".

This situation has certain similarity to the isochoric motions of compressible bodies as compared to the motions of incompressible solids. As is well-known, in the latter case the pressure is not determined through the constitutive equations. Finally, in the indeterminate couple stress theory, the number of boundary conditions on the surface tractions and couples must be reduced from six to five. A consistent set of boundary conditions must not violate the uniqueness theorem. Mindlin and Tiersten [1962] have obtained a uniqueness theorem for the following set of boundary conditions.

Let $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ be a set of orthogonal curvilinear coordinates taken in such a way that $\bar{x}_3 = \bar{x}_3^0$ locally coincides with the surface S of the body. The boundary conditions consist of specifying at $\bar{x}_3 = \bar{x}_3^0$ one factor in each of the five products

$$(23.14) \qquad \bar{p}_1\bar{u}_1 \, , \; \bar{p}_2\bar{u}_2, \; \bar{t}_{(33)}\bar{u}_3 \, , \; \bar{m}_{31}\bar{r}_1 \, , \; \bar{m}_{32}\bar{r}_2$$

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[1] In this regard, see the discussion given in Eringen [1962, Art. 40]. See also Mindlin and Tiersten [1962], Toupin [1962], and Eringen [1964b].

where

$$\bar{p}_1 \equiv \bar{t}_{(31)} - \frac{1}{2}(\bar{m}_{12;i} - \bar{m}_{33;2})$$

(23.15)

$$\bar{p}_2 \equiv \bar{t}_{(32)} + \frac{1}{2}(\bar{m}_{11;i} - \bar{m}_{33;1})$$

Here an index placed after a semicolon denotes directional differentiation along the corresponding curvilinear coordinate, and a superposed bar the boundary values of the quantities involved. Curvilinear components of the displacement vector on S are denoted by $\bar{u}_k$, the couple stress by $\bar{m}_{k\ell}$, and the stress tensor by $\bar{t}_{k\ell}$.

If an edge is an intersection of two orthogonal surfaces $x_3 = \bar{x}_3^0$ and $\bar{x}_1 = \bar{x}_1^0$, then we must also specify

(23.16) $$[\bar{m}_{33}]_{\bar{x}_3^0} - [\bar{m}_{11}]_{\bar{x}_1^0} \quad \text{or} \quad \bar{u}_2$$

The reduction of the number of boundary conditions from six to the above five, (23.15), is similar to the one encountered in the Bernoulli-Euler theory of thin plates. Conditions (23.16) are the analogs to the corner conditions.

Under suitable regularity assumptions, the above five boundary conditions, together with the assignment of the body force field $\rho \underset{\sim}{f}$, curl $(\rho \underset{\sim}{\ell})$, and the initial values of $\underset{\sim}{u}$ and $\dot{\underset{\sim}{u}}$ (with $j \equiv 0$), are sufficient for the unique determination of $e_{k\ell}$, $r_{k,\ell}$, $t_{(k\ell)}$ and $m_{[k\ell]}$. The displacement field $\underset{\sim}{u}$ and the rotation $\underset{\sim}{r}$ are unique too within an arbitrary rigid body displacement field.

The indeterminate couple stress theory described above has many apparent limitations. Whether or not bodies can undergo such constrained motions is not known. The appearance of nonobjective quantities in the constitutive equations, the limitations on the spin inertia and the body couple field, and physically unnatural boundary conditions leaves much to be desired. Experimental comparison for certain practical applications exists (cf. Schijve [1966]). A discussion of the weakness of this theory indicating the disagreement of the theoretical results with the experiments has been given by Kaloni and Ariman [1967]. Nevertheless, recent literature contains a large number of solutions in the field.

24. PROPAGATION OF WAVES IN AN INFINITE MICROPOLAR ELASTIC SOLID[1]

Here and in the following several articles we investigate solutions of certain dynamic and static problems in linear isotropic micropolar elasticity. Essential to these problems are the field equations (22.5) and (22.6) and boundary conditions of the type listed in (22.11) to (22.12).

The propagation of linear isotropic micropolar elastic waves with vanishing body loads is governed by (22.5) and (22.6) or

$$(24.1)\quad (c_1^2 + c_3^2)\nabla\nabla \cdot \mathbf{u} - (c_2^2 + c_3^2)\nabla \times \nabla \times \mathbf{u} + c_3^2\nabla \times \boldsymbol{\phi} = \ddot{\mathbf{u}}$$

$$(24.2)\quad (c_4^2 + c_5^2)\nabla\nabla \cdot \boldsymbol{\phi} - c_4^2\nabla \times \nabla \times \boldsymbol{\phi} + \omega_0^2\nabla \times \mathbf{u} - 2\omega_0^2\boldsymbol{\phi} = \ddot{\boldsymbol{\phi}}$$

where

$$(24.3)\quad \begin{aligned} &c_1^2 \equiv \frac{\lambda+2\mu}{\rho}, \quad c_2^2 \equiv \frac{\mu}{\rho}, \quad c_3^2 \equiv \frac{\kappa}{\rho} \\ &c_4^2 \equiv \frac{\gamma}{\rho j}, \quad c_5^2 \equiv \frac{\alpha+\beta}{\rho j}, \quad \omega_0^2 \equiv \frac{c_3^2}{j} = \frac{\kappa}{\rho j} \end{aligned}$$

We decompose the vectors $\mathbf{u}$ and $\boldsymbol{\phi}$ into scalar and vector potentials as follows:

$$(24.4)\quad \begin{aligned} &\mathbf{u} = \nabla u + \nabla \times \mathbf{U}, \quad \nabla \cdot \mathbf{U} = 0 \\ &\boldsymbol{\phi} = \nabla\phi + \nabla \times \boldsymbol{\Phi}, \quad \nabla \cdot \boldsymbol{\Phi} = 0 \end{aligned}$$

Substituting these into (24.1) and (24.2), we see that these equations are satisfied if

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[1] The present section is based on the work of Parfitt and Eringen [1966]

(24.5) $$(c_1^2 + c_3^2)\nabla^2 u = \ddot{u}$$

(24.6) $$(c_4^2 + c_5^2)\nabla^2\phi - 2\omega_0^2\phi = \ddot{\phi}$$

(24.7) $$(c_2^2 + c_3^2)\nabla^2 \underset{\sim}{U} + c_3^2\underset{\sim}{\nabla} \times \underset{\sim}{\Phi} = \ddot{\underset{\sim}{U}}$$

(24.8) $$c_4^2\nabla^2\underset{\sim}{\Phi} - 2\omega_0^2\underset{\sim}{\Phi} + \omega_0^2\underset{\sim}{\nabla} \times \underset{\sim}{U} = \ddot{\underset{\sim}{\Phi}}$$

It may be observed that (24.5) and (24.6) are uncoupled for the scalar potentials u and ϕ, while equations (24.7) and (24.8) for the vector potentials are coupled.

Plane waves advancing in the positive direction of the unit vector n may be expressed as

(24.9) $$\{u, \phi, \underset{\sim}{U}, \underset{\sim}{\Phi}\} = \{a, b, \underset{\sim}{A}, \underset{\sim}{B}\} \ \exp [ik (\underset{\sim}{n} \cdot \underset{\sim}{r} - vt)]$$

where (a,b) are complex constants, $(\underset{\sim}{A}, \underset{\sim}{B})$ are complex constant vectors, k is the wave number, and $\underset{\sim}{r}$ is the position vector. Thus

(24.10) $$k \equiv \frac{2\pi}{\ell} \quad , \qquad \underset{\sim}{r} = x_k \underset{\sim}{i}_k$$

in which ℓ is the wave length and $\underset{\sim}{i}_k$ are the unit rectangular base vectors.

Substituting (24.9) into (24.5) gives

(24.11) $$v_1^2 = c_1^2 + c_3^2 = (\lambda + 2\mu + \kappa)/\rho$$

which shows that a plane wave with the displacement vector

(24.12) $$\underset{\sim}{u}_1 = ik_1 a\underset{\sim}{n} \ \exp [ik_1(\underset{\sim}{n} \cdot \underset{\sim}{r} - v_1 t)]$$

may exist in the direction of propation $\underset{\sim}{n}$, Fig. 24.1. These waves are the counterpart of the classical irrotational waves and reduce to them when $\kappa = 0$.

We designate these waves, <u>longitudinal displacement waves</u>.

A second scalar plane wave is the solution of (24.6) in the form (24.9). The wave speed in this case is

$$v_2^2 = c_4^2 + c_5^2 + 2\omega_0^2 k^{-2} \tag{24.13}$$

If we introduce the angular frequency ω

$$\omega_i = 2\pi f_i = 2\pi v_i/\ell = kv_i \tag{24.14}$$

the wave speed may be expressed as

$$v_2^2 = (c_4^2 + c_5^2)\left(1 - \frac{2\omega_0^2}{\omega_2^2}\right)^{-1} = \frac{\alpha+\beta+\gamma}{\rho j\left(1 - \frac{2\kappa}{\rho j\omega_2^2}\right)} \tag{24.15}$$

The speed of these waves depends on the frequency. Hence they are dispersive. Since

$$\alpha + \beta + \gamma \geq 0 \tag{24.16}$$

we see that such waves can exist whenever

$$\omega_2 > \sqrt{2}\omega_0 \tag{24.17}$$

These waves will be called <u>longitudinal microrotation waves</u>, Fig. 24.1. The microrotation vector is given by

$$\underset{\sim}{\phi} = \underset{\sim}{\nabla}\phi = ik_2 b\underset{\sim}{n} \exp\left[ik_2(\underset{\sim}{n} \cdot \underset{\sim}{r} - v_2 t)\right] \tag{24.18}$$

For the case of $\omega_2 = \sqrt{2}\omega_0 \equiv \omega_c$, we have $v_2 = \infty$ and the wave does not exist.

When $\omega_2 < \sqrt{2}\omega_0$, v_2 becomes purely imaginary, i.e.,

$$v_2 = \pm i|v_2| \quad , \quad i \equiv \sqrt{-1} \tag{24.19}$$

It may be seen that a standing wave of the form

$$\phi = b \exp\left(-\frac{\omega_2}{|v_2|} \underset{\sim}{n} \cdot \underset{\sim}{r}\right) \exp(-i\omega_2 t) \tag{24.20}$$

is possible.

For such waves, propagation is possible if $\omega_2 > \sqrt{2}\omega_0$. Hence, $\sqrt{2}\omega_0 \equiv \omega_c$ is a cut-off frequency for these waves.

The vector wave solutions are obtained by substituting (24.9) into (24.7) and (24.8). This results in two simultaneous vector equations for the unknowns $\underset{\sim}{A}$ and $\underset{\sim}{B}$.

$$\begin{aligned} \alpha_A \underset{\sim}{A} + i\alpha_B \underset{\sim}{n} \times \underset{\sim}{B} &= \underset{\sim}{0} \\ i\beta_A \underset{\sim}{n} \times \underset{\sim}{A} + \beta_B \underset{\sim}{B} &= \underset{\sim}{0} \end{aligned} \tag{24.21}$$

which for nonvanishing α_A, α_B, β_A, and β_B are subject to

$$\underset{\sim}{n} \cdot \underset{\sim}{A} = 0 \quad , \quad \underset{\sim}{n} \cdot \underset{\sim}{B} = 0 \tag{24.22}$$

resulting from $(24.4)_2$ and $(24.4)_4$. Here

$$\begin{aligned} \alpha_A &\equiv k^2(v^2 - c_2^2 - c_3^2) \quad , \quad \alpha_B \equiv kc_3^2 \\ \beta_A &\equiv k\omega_0^2 \quad , \quad \beta_B \equiv k^2(v^2 - c_4^2 - 2\omega_0^2 k^{-2}) \end{aligned} \tag{24.23}$$

Equations (24.22) show that the vectors $\underset{\sim}{A}$ and $\underset{\sim}{B}$ lie in a common plane whose unit normal is $\underset{\sim}{n}$. Solving from $(24.21)_2$ for $\underset{\sim}{B}$ we have

$$\underset{\sim}{B} = -i \frac{\beta_A}{\beta_B} \underset{\sim}{n} \times \underset{\sim}{A} \tag{24.24}$$

Hence, the three vectors $\underset{\sim}{n}$, $\underset{\sim}{A}$, and $\underset{\sim}{B}$ are mutually perpendicular. Moreover, vanishing $\underset{\sim}{A}$ implies vanishing $\underset{\sim}{B}$. These two types of waves are, therefore, coupled and cannot exist without each other. From $(24.4)_1$ and $(24.4)_3$ we see that $\underset{\sim}{u}$ and $\underset{\sim}{\phi}$ corresponding to $\underset{\sim}{U}$ and $\underset{\sim}{\Phi}$ are normal to each other and to the direction of propagation $\underset{\sim}{n}$. Hence these waves are transverse waves. We call the waves that are associated with $\underset{\sim}{U}$, <u>transverse displacement waves</u>, and those that are associated with $\underset{\sim}{\Phi}$, <u>transverse microrotation waves</u>, Fig. 24.2. The transverse displacement waves have their classical analogues in equivoluminal waves and in the limit they reduce to these waves.

The velocities of propagation of these waves are determined by carrying (24.24) into $(24.21)_2$ and using $(24.22)_1$. This gives

(24.25) $$av^4 + bv^2 + c = 0$$

where

(24.26)
$$\begin{aligned} a &\equiv 1 - 2\omega_0^2\omega^{-2} \\ b &\equiv -[c_4^2 + c_2^2(1 - 2\omega_0^2\omega^{-2}) + c_3^2(1 - \omega_0^2\omega^{-2})] \\ c &\equiv c_4^2(c_2^2 + c_3^2) \end{aligned}$$

The positive real roots of (24.25) are

(24.27)
$$\begin{aligned} v_3 &= [\frac{1}{2a}(-b + \sqrt{b^2-4ac})]^{1/2} \\ v_4 &= [\frac{1}{2a}(-b - \sqrt{b^2-4ac})]^{1/2} \end{aligned}$$

By studying the discriminant

$$\sqrt{b^2-4ac} = \{[c_4^2 - c_2^2 - c_3^2 + 2(c_2^2 + \frac{1}{2}c_3^2)\omega_0^2\omega^{-2}]^2$$
$$+ 4c_3^2c_4^2\omega_0^2\omega^{-2}\}^{1/2}$$

under the conditions $\kappa \geq 0$, $\gamma \geq 0$ compatible with (21.1), we find that v_3 is real when $\omega > \omega_c$ and v_4 is real for all values of ω. The frequency $\omega_c \equiv \sqrt{2}\omega_0$ is again a cut-off frequency for waves propagating with velocity v_3.

In summary, we found that in an infinite micropolar elastic solid six different types of plane waves propagating with four distinct speeds of propagation can exist. These are:

(a) A longitudinal displacement wave propagating with speed v_1.

(b) A longitudinal microrotation wave with speed v_2 propagating in the longitudinal direction whenever the frequency of these waves is above the cut-off frequency ω_c. These two types of waves are uncoupled.

(c) Two sets of coupled transverse displacement waves and transverse microtation waves at speeds v_3 and v_4. Of these, the waves having velocity v_3 can exist when $\omega > \omega_c$ otherwise they degenerate into distance decaying sinusoidal vibrations.

A detailed analysis of the wave speeds v_2, v_3, and v_4 is sketched in Figs. (24.3) and (24.4).

By use of (21.1), Parfitt and Eringen [1966] have shown that a consistent solution for v_3 and v_4 requires the existence of

$$c_4^2 \geq c_2^2 + c_3^2$$

or the additional inequality

$$\frac{\gamma}{j} \geq \mu + \kappa \tag{24.28}$$

In addition, the wave speeds as a function of ω must satisfy

$$v_4^2(0) \leq v_4^2(\omega_c) \leq v_4^2(\infty) \tag{24.29}$$

$$v_2^2(\infty) > v_3^2(\infty) > v_4^2(\infty) \tag{24.30}$$

Also they found that

$$v_1^2 > v_4^2 \tag{24.31}$$

The study of the relative magnitudes of v_2, v_3, and v_4 requires knowledge of the relative magnitude of the sum of the constitutive coefficients $\alpha + \beta$ with respect to $j\kappa$. Thus

$$\begin{aligned} &v_2^2 > v_3^2 > v_4^2 && \text{for } \omega_c < \omega && , \quad \alpha + \beta > \tfrac{1}{2}\, jk \\ &v_3^2 \geq v_2^2 \geq v_4^2 && \text{for } \omega_c < \omega \leq \omega^* && , \quad \alpha + \beta < \tfrac{1}{2}\, jk \\ &v_2^2 \geq v_3^2 \geq v_4^2 && \text{for } \omega^* < \omega && \end{aligned} \tag{24.32}$$

where ω^* is a solution of $v_2^2(\omega^*) = v_3^2(\omega^*)$ in the range $\omega_c < \omega^* < \infty$. For other details the reader is referred to Parfitt and Eringen [1966].

25. REFLECTION OF A LONGITUDINAL DISPLACEMENT WAVE

In this section we study the reflection of a plane longitudinal displacement wave at a stress-free plane surface of a micropolar half space. If $x = 0$ is the plane of the incident displacement vector, then the reflected waves from the boundary surface $z = 0$ can be shown to remain in this plane so that the problem is a two-dimensional one. On the boundary plane $z = 0$ being free from traction we must have $\underset{\sim}{t}_{(\underset{\sim}{n})} = \underset{\sim}{m}_{(\underset{\sim}{n})} = \underset{\sim}{0}$. Since we also have $u = \phi_2 = \phi_3 = 0$ through (22.11) and (22.12) for the tractions on $z = 0$, we will have $t_{(z)z} = t_{zz}$, $t_{(z)y} = t_{zy}$ and $m_{(z)x} = m_{zx}$, or using the constitutive equations and (24.4) we have

$$t_{zz} = \lambda(u_{,yy} + u_{,zz}) + (2\mu + \kappa)(u_{,zz} - U_{x,yz}) = 0$$

$$t_{zy} = \mu(u_{,yz} - U_{x,yy}) + (\mu + \kappa)(u_{,yz} + U_{x,zz}) + \kappa(\Phi_{z,y} - \Phi_{y,z}) = 0 \tag{25.1}$$

$$m_{zx} = \gamma(\Phi_{z,yz} - \Phi_{y,zz}) = 0$$

The nonvanishing components of the solution vectors (24.9) appropriate to this problem are given by

$$u_\alpha = a_\alpha \exp\,[i(k_1 \underset{\sim}{n}_\alpha \cdot \underset{\sim}{r} - \omega_1 t)]$$

$$U_\beta = \underset{\sim}{i} A_{\beta x} \exp\,[i(k_\beta \underset{\sim}{n}_\beta \cdot \underset{\sim}{r} - \omega_\beta t)] \tag{25.2}$$

$$\Phi_\beta = (B_{\beta y}\underset{\sim}{j} + B_{\beta z}\underset{\sim}{k}) \exp\,[i(k_\beta \underset{\sim}{n}_\beta \cdot \underset{\sim}{r} - \omega_\beta t)]$$

where $\omega_\beta = k_\beta v_\beta (\alpha = 1,2; \beta = 3,4)$ and the repeated indices are not summed. The coefficients $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are related to each other by

$$\underset{\sim}{B}_3 = -\frac{i\omega_0^2 A_{3x}}{k_3(v_3^2 - 2\omega_0^2 k_3^{-2} - c_4^2)} (n_{3z}\underset{\sim}{j} - n_{3y}\underset{\sim}{k}) \tag{25.3}$$

with a similar equation for B_4.

The potentials listed in (25.2) satisfy the boundary conditions $(25.1)_1$ at z = 0 if

$$\omega_1 = \omega_3 = \omega_4 = \omega$$

$$k_1 n_{1y} = k_1 n_{2y} = k_3 n_{3y} = k_4 n_{4y} \tag{25.4}$$

$$k_1 n_{1x} = k_1 n_{2x} = k_3 n_{3x} = k_4 n_{4x}$$

and

$$[\lambda k_1^2 + (2\mu + \kappa)k_1^2 n_{1z}^2]a_1 + [\lambda k_1^2 + (2\mu + \kappa)k_1^2 n_{2z}^2]a_2 \tag{25.5}$$

$$- (2\mu + \kappa)k_3^2 n_{3y} n_{3z} A_{3x} - (2\mu + \kappa)k_4^2 n_{4y} n_{4z} A_{4x} = 0$$

Since the incident wave is in the x = 0 plane, $n_{1x} = 0$ and $(25.4)_3$ yields

$$n_{2x} = n_{3x} = n_{4x} = 0$$

which is the proof of our statement that all reflected waves are on the x = 0 plane. Writing

$$n_{1y} \equiv \cos\theta_1 \quad , \quad n_{2y} \equiv \cos\theta_2 \quad , \quad n_{3y} \equiv \cos\theta_3 \quad , \quad n_{4y} \equiv \cos\theta_4$$

(see Fig. 25.1) and $\omega_i = k_i v_i$ from (25.4) we get

$$\cos\theta_2 = \cos\theta_1 \ , \quad \cos\theta_3 = \frac{v_3}{v_1}\cos\theta_1 \ , \quad \cos\theta_4 = \frac{v_4}{v_1}\cos\theta_1 \tag{25.6}$$

where v_i are the speeds of the various waves found in Art. 24. From (25.6) it is clear that $\theta_1 = \theta_2$.

Substituting (25.2) into the remaining two equations of (25.1), we obtain two other equations. These two equations and (25.5) are adequate to determine the amplitude ratios a_2/a_1, A_{3x}/a_1 and A_{4x}/a_1. The amplitude $\underset{\sim}{B}$ is determined from equations (25.3) and similar ones for $\underset{\sim}{B}_4$.

If we set $n_{1y} = 0$ in (25.5), we find that $a_2 = -a_1$. This shows that for normal incidence the reflected longitudinal displacement wave is also normal to the boundary. For $n_{1y} \neq 0$, the following are the solutions for the ratios of the wave amplitudes

$$(25.7)\quad \frac{a_2}{a_1} = \{[\lambda + (\lambda+2\mu+\kappa)\tan^2\theta_1][(\mu+\kappa)\tan^2\theta_4 - \mu - (\mu+\kappa)Q_1^2 \tan\theta_3 \tan\theta_4 - Q_2^2$$

$$+ (\mu Q_1^2+Q_3^2)\frac{\tan\theta_4}{\tan\theta_3}] + (2\mu+\kappa)^2 \tan\theta_1 \tan\theta_4(Q_1^2-1)\}$$

$$\times\{-[\lambda + (\lambda+2\mu+\kappa)\tan^2\theta_1][(\mu+\kappa)\tan^2\theta_4 - \mu - (\mu+\kappa)Q_1^2 \tan\theta_3 \tan\theta_4 - Q_2^2$$

$$+ (\mu Q_1^2+Q_3^2)\frac{\tan\theta_4}{\tan\theta_3}] + (2\mu+\kappa)^2(Q_1^2-1)\tan\theta_1 \tan\theta_4\}^{-1}$$

$$(25.8)\quad \frac{A_{4x}}{a_1} = -2(2\mu+\kappa)[\lambda + (\lambda+2\mu+\kappa)\tan^2\theta_1]\tan\theta_1$$

$$\times\{-[\lambda + (\lambda+2\mu+\kappa)\tan^2\theta_1][(\mu+\kappa)\tan^2\theta_4 - \mu - (\mu+\kappa)Q_1^2 \tan\theta_3 \tan\theta_4$$

$$- Q_2^2 + (\mu Q_1^2+Q_3^2)\frac{\tan\theta_4}{\tan\theta_3}] + (2\mu+\kappa)^2(Q_1^2-1)\tan\theta_1 \tan\theta_4\}^{-1}$$

$$(25.9)\quad Q_1^2 \equiv [v_3^2(1 - \frac{2\omega_0^2}{\omega^2}) - c_4^2][v_4^2(1 - \frac{2\omega_0^2}{\omega^2}) - c_4^2]^{-1}$$

$$Q_2^2 \equiv \frac{\omega_0^2}{\omega^2}\frac{v_4^2}{v_{4y}^2}[v_4^2(1 - \frac{2\omega_0^2}{\omega^2}) - c_4^2]^{-1}$$

$$Q_3^2 \equiv \frac{\omega_0^2}{\omega^2}\frac{v_3^2}{v_{3y}^2}[v_4^2(1 - \frac{2\omega_0^2}{\omega^2}) - c_4^2]^{-1}$$

These results were studies in detail by Parfitt and Eringen [1966] for various special cases. A summary of their findings is given below:

A longitudinal displacement wave at a plane stress free boundary in general produces three reflected waves (as compared with the two waves of the classical theory); one longitudinal displacement wave reflected at an angle which is the same as the angle of incidence, and two sets of coupled transverse waves, one travelling with speed v_3 and the other set with speed v_4 (Fig. 25.1). Their angles of reflection are calculated by (25.6). For normal incidence ($\theta_1 = 90°$) the waves at speeds v_3 and v_4 vanish and the reflected wave is a longitudinal displacement wave normal to the boundary. The amplitude ratios as functions of the angle of incidence θ_1 are given by (25.7) to (25.9) and similar equations for B_4.

The general solution prevails as θ_1 decreases from 90° to a critical angle θ_1^* which makes $\theta_3 = 0$ and $A_{4x} = 0$. At this angle of incidence we have a surface motion travelling at speed v_3 and a reflected longitudinal wave at speed v_1 and angle $\theta_2 = \theta_1 = \theta_1^*$. As θ_1 decreases from θ_1^* to zero, the angle θ_3 becomes complex. The interpretation in this case is that a longitudinal wave is reflected into the medium at an angle θ_1 and a coupled transverse wave (decaying with depth and travelling at a speed c, in the range $v_1 \leq c \leq v_3$, is reflected. For $\theta_1 \to 0$, a limit solution is possible

Parfitt and Eringen have also studied the reflection of coupled transverse shear and microrotational waves and the reflection of a longitudinal microrotational wave. For these and other interesting results, the reader is referred to the foregoing reference.

26. MICROPOLAR SURFACE WAVES[1]

In this section we investigate the propagation of surface waves in a micropolar half space. In Section 25 we have shown that the incident and reflected waves propagate in the same plane. We select this plane to be the x = 0 plane. Thus we take $u_1 = \phi_2 = \phi_3 = 0$, $\underset{\sim}{f} = \underset{\sim}{\ell} = \underset{\sim}{0}$, and $u_2 \equiv v(y,z,t)$ $u_3 \equiv w(y,z,t)$ and $\phi_1 \equiv \phi(y,z,t)$ as functions of y, z, and t only. In this case (22.1) and (22.2) reduce to

$$(\lambda + \mu) \frac{\partial}{\partial y} (\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) + (\mu + \kappa)\nabla^2 v + \kappa \frac{\partial \phi}{\partial z} - \rho \frac{\partial^2 v}{\partial t^2} = 0$$

$$(26.1) \qquad (\lambda + \mu) \frac{\partial}{\partial z} (\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) + (\mu + \kappa)\nabla^2 w - \kappa \frac{\partial \phi}{\partial y} - \rho \frac{\partial^2 w}{\partial t^2} = 0$$

$$\gamma\nabla^2\phi + \kappa(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}) - 2\kappa\phi - \rho j \frac{\partial^2 \phi}{\partial t^2} = 0$$

where ∇^2 is the two-dimensional laplacian operator, e.g.,

$$\nabla^2 \equiv \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

We consider waves which are propagating in the plane x = 0 with an amplitude decay in the z direction.

$$v = A \exp(-\zeta z) \exp[iq(y - ct)]$$

$$(26.2) \qquad w = B \exp(-\zeta z) \exp[iq(y - ct)]$$

$$\phi = C \exp(-\zeta z) \exp[iq(y - ct)]$$

---

[1] Suhubi and Eringen [1964b]

Substituting (26.2) into (26.1) gives three equations

$$[-(\lambda + 2\mu + \kappa)q^2 + (\mu + \kappa)\zeta^2 + \rho q^2 c^2]A - (\lambda + \mu)iq\zeta B - \kappa\zeta C = 0$$

$$(26.3) \quad -(\lambda + \mu)iq\zeta A + [(\lambda + 2\mu + \kappa)\zeta^2 - (\mu + \kappa)q^2 + \rho q^2 c^2]B - iq\kappa C = 0$$

$$\kappa\zeta A + iq\kappa B + [\gamma(\zeta^2 - q^2) - 2\kappa + \rho j q^2 c^2]C = 0$$

A nonvanishing solution for A, B, and C may exist if the determinant of the coefficients is zero. This gives

$$(26.4) \quad [(\varepsilon + \frac{c_1^2}{c_2^2})\zeta^2 - (\varepsilon + \frac{c_1^2 - c^2}{c_2^2})q^2]\{[j\theta\zeta^2 - j(\theta - \frac{c^2}{c_2^2})q^2 - 2\varepsilon] \times [\varepsilon + 1)\zeta^2 - (\varepsilon + 1 - \frac{c^2}{c_2^2})q^2] + \varepsilon^2(\zeta^2 - q^2)\} = 0$$

where

$$(26.5) \qquad \varepsilon \equiv c_3^2/c_2^2 \ , \quad \theta \equiv c_4^2/c_2^2$$

and c_1 to c_5 are given by (24.3). A set of approximate roots of (26.4) is obtained by neglecting the terms containing ε^2. Hence

$$\zeta_1^2 = (1 - \frac{c^2}{c_1^2 + c_3^2})q^2$$

$$(26.6) \qquad \zeta_2^2 = [1 - (1 - \varepsilon)\frac{c^2}{c_2^2}]q^2$$

$$\zeta_3^2 = 2\frac{\varepsilon}{j\theta} + (1 - \frac{c^2}{c_4^2})q^2$$

In order for the waves to be surface waves, we must consider only the positive values of the roots ζ_1, ζ_2, and ζ_3. The displacement and microrotation field can now be expressed as

$$v = \sum_{k=1}^{3} A_k \exp(-\zeta_k z) \exp[iq(y - ct)]$$

(26.7) $$w = \sum_{k=1}^{3} \lambda_k A_k \exp(-\zeta_k z) \exp[iq(y - ct)]$$

$$\phi = \mu_3 A_3 \exp(-\zeta_3 z) \exp[iq(y - ct)]$$

where

(26.8) $$\lambda_1 = \frac{i\zeta_1}{q}, \quad \lambda_2 = \frac{iq}{\zeta_2},$$

$$\lambda_3 = \frac{iq}{\zeta_3}, \quad \mu_3 = \frac{1}{\varepsilon\zeta_3}\left[(1 + \varepsilon)\left(\frac{2\varepsilon}{I_0\theta} - \frac{c^2}{c_4^2} q^2\right) + \frac{c^2}{c_2^2} q^2\right]$$

On stress-free boundary surface we must have

(26.9) $$\left.\begin{aligned} t_{zz} &= \lambda \frac{\partial v}{\partial y} + (\lambda + 2\mu + \kappa) \frac{\partial w}{\partial z} = 0 \\ t_{zy} &= \mu \frac{\partial w}{\partial y} + (\mu + \kappa) \frac{\partial v}{\partial z} + \kappa\phi = 0 \\ m_{zx} &= \gamma \frac{\partial \phi}{\partial z} = 0 \end{aligned}\right\} \quad \text{at } z = 0$$

Substituting (26.7) into (26.9), we obtain a set of three homogeneous equations for A_1, A_2, and A_3. The determinant of the coefficients must vanish. Hence

(26.10) $$\omega^3 - 8\omega^2 + 8(3 - k)\omega - 16(1 - k) - 16\varepsilon(1 - k\omega) = 0$$

(26.11) $$\zeta_3\mu_3 = 0$$

where

(26.12) $$\omega \equiv c^2/c_2^2, \quad k \equiv c_2^2/c_1^2$$

For $\varepsilon = 0$, (26.10) reduces to the classical expression of the Rayleigh surface waves. Denoting the values of ω for this case, with ω_R to a first-order approximation in ε, we obtain

$$\omega = \omega_R + \frac{16(1-k\omega_R)}{3\omega_R^2 - 16\omega_R + 8(3-2k)}\,\varepsilon \tag{26.13}$$

For k = 1/3, which correspond to Poissons ratio 1/4, and for the incompressible solids (k = 0) we have, respectively,

$$\begin{aligned} c &= 0.919\ (1 + 0.932\varepsilon)c_2\ , \quad (k = 1/3) \\ c &= 0.955\ (1 + 0.783\varepsilon)c_2\ , \quad (k = 0) \end{aligned} \tag{26.14}$$

The terms containing ε are the first-order corrections to the Rayleigh wave velocity in each case.

Equation (26.11) gives the speed of propagation of a new type surface wave not encountered in classical elasticity. This is given by

$$\frac{c}{c_2} = [2\varepsilon/j(1 + \varepsilon - \theta)]^{1/2} q^{-1} \simeq (2\varepsilon/j)^{1/2} q^{-1} \tag{26.15}$$

This new wave speed depends on $\sqrt{\varepsilon}$ and it is dispersive.

27. STRESS CONCENTRATION AROUND A CIRCULAR HOLE

For the determination of the stresses in a plate with a circular hole, we need to express the basic equations of micropolar elasticity in plane polar coordinates (r, θ), Fig. 27.1. For the equations of balance of momenta, parallel to classical elasticity, we have

$$
\begin{aligned}
&\frac{\partial t_{rr}}{\partial r} + \frac{1}{r}\frac{\partial t_{\theta r}}{\partial \theta} + \frac{t_{rr} - t_{\theta\theta}}{r} + \rho f_r = \rho \frac{\partial^2 u_r}{\partial t^2} \\
(27.1)\qquad &\frac{\partial t_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial t_{\theta\theta}}{\partial \theta} + \frac{t_{r\theta} + t_{\theta r}}{r} + \rho f_\theta = \rho \frac{\partial^2 u_\theta}{\partial t^2} \\
&\frac{\partial m_{rz}}{\partial r} + \frac{1}{r}\frac{\partial m_{\theta z}}{\partial \theta} + \frac{m_{rz}}{r} + t_{r\theta} - t_{\theta r} + \rho \ell_z = \rho j \frac{\partial^2 \phi_z}{\partial t^2}
\end{aligned}
$$

For the nonzero components of the strain tensor $\varepsilon_{k\ell}$ given by (4.17) through the methods presented in Eringen [1962, Appendix] we obtain

$$
\begin{aligned}
&\varepsilon_{rr} = \frac{\partial u_r}{\partial r} \\
&\varepsilon_{\theta\theta} = \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right) \\
(27.2)\qquad &\varepsilon_{r\theta} = \frac{\partial u_\theta}{\partial r} - \phi_z \\
&\varepsilon_{\theta r} = \frac{1}{r}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\right) + \phi_z
\end{aligned}
$$

The constitutive equations for the stress read

$$
\begin{aligned}
&t_{rr} = (\lambda + 2\mu + \kappa)\varepsilon_{rr} + \lambda\varepsilon_{\theta\theta} \\
&t_{\theta\theta} = \lambda\varepsilon_{rr} + (\lambda + 2\mu + \kappa)\varepsilon_{\theta\theta} \\
(27.3)\qquad &t_{r\theta} = (\mu + \kappa)\varepsilon_{r\theta} + \mu\varepsilon_{\theta r} \\
&t_{\theta r} = \mu\varepsilon_{r\theta} + (\mu + \kappa)\varepsilon_{\theta r}
\end{aligned}
$$

In these equations (t_{rr}, $t_{r\theta}$,...), (ε_{rr}, $\varepsilon_{r\theta}$,...) and u_r and u_θ are, respectively, the physical components of the stresses, strains, and displacements.

The equations of compatibility (8.9) in plane polar coordinates take the form

$$(27.4)\qquad \begin{aligned}
&\frac{\partial \varepsilon_{\theta r}}{\partial r} + \frac{\varepsilon_{\theta r} + \varepsilon_{r\theta}}{r} - \frac{1}{r}\frac{\partial \varepsilon_{rr}}{\partial \theta} - \frac{\partial \phi_z}{\partial r} = 0 \\
&\frac{\partial \varepsilon_{\theta\theta}}{\partial r} + \frac{\varepsilon_{\theta\theta} - \varepsilon_{rr}}{r} - \frac{1}{r}\frac{\partial \varepsilon_{r\theta}}{\partial \theta} - \frac{1}{r}\frac{\partial \phi_z}{\partial \theta} = 0 \\
&\frac{\partial m_{\theta z}}{\partial r} + \frac{m_{\theta z}}{r} - \frac{1}{r}\frac{\partial m_{rz}}{\partial \theta} = 0
\end{aligned}$$

For static problems and vanishing body loads upon introducing the stress functions F(r, θ) and G(r, θ) by

$$(27.5)\qquad \begin{aligned}
t_{rr} &= \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} - \frac{1}{r}\frac{\partial^2 G}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial G}{\partial \theta} \\
t_{\theta\theta} &= \frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial^2 G}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial G}{\partial \theta} \\
t_{r\theta} &= -\frac{1}{r}\frac{\partial^2 F}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial F}{\partial \theta} - \frac{1}{r}\frac{\partial G}{\partial r} - \frac{1}{r^2}\frac{\partial^2 G}{\partial \theta^2} \\
t_{\theta r} &= -\frac{1}{r}\frac{\partial^2 F}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial F}{\partial \theta} + \frac{\partial^2 G}{\partial r^2} \\
m_{rz} &= \frac{\partial G}{\partial r}, \quad m_{\theta z} = \frac{1}{r}\frac{\partial G}{\partial \theta}
\end{aligned}$$

we see that equations (27.1) are satisfied identically. From (27.3), we solve for the strains in terms of the stresses and then substitute (27.5) for the stress components. If we now use the expressions (27.4), we get

$$\frac{\partial}{\partial r}(G - c^2\nabla^2 G) = -2(1 - \nu)b^2 \frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2 F)$$

(27.6)

$$\frac{1}{r}\frac{\partial}{\partial\theta}(G - c^2\nabla^2 G) = 2(1 - \nu)b^2 \frac{\partial}{\partial r}(\nabla^2 F)$$

where

$$c^2 \equiv \frac{\gamma(\mu + \kappa)}{\kappa(2\mu + \kappa)} \quad , \quad b^2 \equiv \frac{\gamma}{2(2\mu + \kappa)}$$

(27.7)

$$\nu \equiv \frac{\lambda}{2\lambda + 2\mu + \kappa} \quad , \quad \nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

Solutions of the following equations are also solutions of (27.6)

(27.8) $$\nabla^2 F = 0 \quad , \quad \nabla^2(G - c^2\nabla^2 G) = 0$$

For the problem of a circular hole in a plate (or cylindrical cavity in an infinite solid) subject to a field of simple tension at infinity, an appropriate solution of (27.8) is

$$F = \frac{T}{4} r^2(1 - \cos 2\theta) + A_1 \log r + \left(\frac{A_2}{r^2} + A_3\right) \cos 2\theta$$

(27.9)

$$G = \left[\frac{A_4}{r^2} + A_5 K_2(r/c)\right] \cos 2\theta$$

where K_2 is the modified Bessel function of the second kind and second order. This set satisfies (27.6) if

(27.10) $$A_4 = 8(1 - \nu)c^2 A_3$$

The remaining four constants A_1, A_2, A_3, and A_5 are to be determined from the boundary conditions

$$t_{rr} = t_{r\theta} = m_{rz} = 0 \quad \text{for } r = a$$

$$(27.11) \qquad \left.\begin{aligned} t_{rr} &= \frac{T}{2}(1 + \cos 2\theta) \\ t_{r\theta} &= -\frac{T}{2}\sin 2\theta \\ m_{rz} &= 0 \end{aligned}\right\} \quad \text{for } r = \infty$$

Here T is the constant tension field at a plane x = const at infinity, Fig. 27.1. By use of (27.5) and (27.9), we calculate the components of the stress and couple stress tensors

$$t_{rr} = \frac{T}{2}(1 + \cos 2\theta) + \frac{A_1}{r^2} - \left(\frac{6A_2}{r^4} + \frac{4A_3}{r^2} - \frac{6A_4}{r^4}\right)\cos 2\theta$$

$$+ \frac{2A_5}{cr}\left[\frac{3c}{r}K_0(r/c) + \left(1 + \frac{6c^2}{r^2}\right)K_1(r/c)\right]\cos 2\theta$$

$$t_{\theta\theta} = \frac{T}{2}(1 - \cos 2\theta) - \frac{A_1}{r^2} + \left(\frac{6A_2}{r^4} - \frac{6A_4}{r^4}\right)\cos 2\theta$$

$$- \frac{2A_5}{cr}\left[\frac{3c}{r}K_0(r/c) + \left(1 + \frac{6c^2}{r^2}\right)K_1(r/c)\right]\cos 2\theta$$

$$(27.12) \quad t_{r\theta} = -\left(\frac{T}{2} + \frac{6A_2}{r^4} + \frac{2A_3}{r^2} - \frac{6A_4}{r^4}\right)\sin 2\theta$$

$$+ \frac{A_5}{cr}\left[\frac{6c}{r}K_0(r/c) + \left(1 + \frac{12c^2}{r^2}\right)K_1(r/c)\right]\sin 2\theta$$

$$t_{\theta r} = -\left(\frac{T}{2} + \frac{6A_2}{r^4} + \frac{2A_3}{r^2} - \frac{6A_4}{r^4}\right)\sin 2\theta$$

$$+ \frac{A_5}{c^2}\left[\left(1 + \frac{6c^2}{r^2}\right)K_0(r/c) + \left(\frac{3c}{r} + \frac{12c^3}{r^3}\right)K_1(r/c)\right]\sin 2\theta$$

$$m_{rz} = -\frac{2A_4}{r^3}\sin 2\theta - \frac{A_5}{c}\left[\frac{2c}{r}K_0(r/c) + \left(1 + \frac{4c^2}{r^2}\right)K_1(r/c)\right]\sin 2\theta$$

$$m_{\theta z} = \left\{\frac{2A_4}{r^3} + \frac{2A_5}{r}\left[K_0(r/c) + \frac{2c}{r}K_1(r/c)\right]\right\}\cos 2\theta$$

Using the boundary conditions (27.11), we find that

$$A_1 = -\frac{T}{2}a^2 \quad , \quad A_2 = \frac{Ta^4(1 - F_1)}{4(1 + F_1)}$$

$$A_3 = \frac{Ta^2}{2(1 + F_1)} \qquad A_4 = \frac{4(1 - \nu)a^2b^2T}{1 + F_1} \tag{27.13}$$

$$A_5 = -\frac{TacF_1}{(1 + F_1)K_1(a/c)}$$

where

$$F_1 \equiv 8(1 - \nu)\frac{b^2}{c^2}\left[4 + \frac{a^2}{c^2} + \frac{2a}{c}\frac{K_0(a/c)}{K_1(a/c)}\right]^{-1} \tag{27.14}$$

Substituting (27.13) into (27.12), we obtain the stress and couple stress fields. The value of $t_{\theta\theta}$ at the perifery of the circular hole is of great interest. For this we obtain

$$t_{\theta\theta} = T\left(1 + \frac{2\cos 2\theta}{1 + F_1}\right) \tag{27.15}$$

The maximum value of this $t_{\theta\theta max}$ occurs at $\theta = \pm\pi/2$.

$$\frac{t_{\theta\theta\ max}}{T} \equiv S_c = \frac{3 + F_1}{1 + F_1} \tag{27.16}$$

The quantity S_c so defined is the stress concentration factor. From (27.14) it is clear that S_c depends on ν, a, b, and c.

The above result, (27.16), was given by Kaloni and Ariman [1967] who adopted the solution of the same problem for the indeterminate couple stress theory given by Mindlin and Tiersten [1962]. If we set $b^2/c^2 = 1$, we obtain for F_1 the one given by Mindlin [1963], namely,

$$F_0 = b(1 - \nu)\left[4 + \frac{a^2}{\ell^2} + \frac{2a}{\ell}\,\frac{K_0(a/\ell)}{K_1(a/\ell)}\right]^{-1} \tag{27.17}$$

where ℓ is a physical length for a given material. The case of $b^2/c^2 = 1$ gives $\kappa = -2\mu$ which is not acceptable on grounds of uniqueness and physical reasonableness since κ cannot be as great as twice the shear modulus. Other discussion of these results and comparison of the stress concentration factors of the indeterminate couple stress theory and micropolar theory are to be found in Kaloni and Ariman [1967]. Below we reproduce several of their curves.

28. GALERKIN AND PAPKOVITCH REPRESENTATIONS[1]

A useful mathematical representation for the displacement and micro-rotation fields can be constructed as follows: Let

$$X_i \equiv \frac{\partial}{\partial x_i}\,,\ T \equiv \frac{\partial}{\partial t}\,,\ Q \equiv X_1^2 + X_2^2 + X_3^2\,,\ Q_1 \equiv (\lambda + 2\mu + \kappa)Q - \rho T^2$$

(28.1)

$$Q_2 \equiv (\mu + \kappa)Q - \rho T^2, Q_3 \equiv (\alpha+\beta+\gamma)Q - \rho j T^2 - 2\kappa,\ Q_4 \equiv \gamma Q - \rho j T^2 - 2\kappa$$

and let the matrix $\underset{\sim}{L} = [L_{ij}]$, $(i,j = 1,2,\ldots,6)$ be given by

(28.2)

$$\underset{\sim}{L} = \begin{bmatrix} Q_2 + (\lambda+\mu)X_1^2 & (\lambda+\mu)X_1X_2 & (\lambda+\mu)X_1X_3 & 0 & -\kappa X_3 & \kappa X_2 \\ (\lambda+\mu)X_1X_2 & Q_2 + (\lambda+\mu)X_2^2 & (\lambda+\mu)X_2X_3 & \kappa X_3 & 0 & -\kappa X_1 \\ (\lambda+\mu)X_1X_3 & (\lambda+\mu)X_2X_3 & Q_2 + (\lambda+\mu)X_3^2 & -\kappa X_2 & \kappa X_1 & 0 \\ 0 & -\kappa X_3 & \kappa X_2 & Q_4 + (\alpha+\beta)X_1^2 & (\alpha+\beta)X_1X_2 & (\alpha+\beta)X_1X_3 \\ \kappa X_3 & 0 & -\kappa X_1 & (\alpha+\beta)X_1X_2 & Q_4 + (\alpha+\beta)X_2^2 & (\alpha+\beta)X_2X_3 \\ -\kappa X_2 & \kappa X_1 & 0 & (\alpha+\beta)X_1X_3 & (\alpha+\beta)X_2X_3 & Q_4+(\alpha+\beta)X_3^2 \end{bmatrix}$$

The equations (22.1) and (22.2) can be expressed in the matrix form

---

[1] Sandru [1966]

$$
L \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = -\rho \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \ell_1 \\ \ell_2 \\ \ell_3 \end{bmatrix} \tag{28.3}
$$

For the inverse L_{ij}^{-1} of the matrix L_{ij} we have, formally,

$$
L_{ij}^{-1} = \frac{N_{ij}}{Q_1 Q_3 (Q_2 Q_4 + \kappa^2 Q)} \tag{28.4}
$$

where

$$
\begin{aligned}
N_{ii} &= Q_3\{Q_1 Q_4 - [(\lambda+\mu)Q_4 - \kappa^2]X_i^2\} \quad , (i = 1,2,3) \\
N_{ii} &= Q_1\{Q_2 Q_3 - [(\alpha+\beta)Q_2 - \kappa^2]X_{i-3}^2\}, \ (i = 4,5,6) \\
N_{ij} &= N_{ji} = Q_3[-(\lambda+\mu)Q_4 + \kappa^2]X_i X_j \quad , i \neq j, \ (i,j = 1,2,3) \\
N_{ij} &= N_{ji} = Q_1[-(\alpha+\beta)Q_2 + \kappa^2]X_{i-3}X_{j-3}, \ i \neq j, \ (i,j = 4,5,6) \\
N_{14} &= N_{25} = N_{36} = N_{41} = N_{52} + N_{63} = 0 \\
N_{15} &= -N_{51} = N_{42} = -N_{24} = \kappa Q_1 Q_3 X_3 \\
N_{16} &= -N_{61} = N_{43} = -N_{34} = -\kappa Q_1 Q_3 X_2 \\
N_{26} &= -N_{62} = N_{53} = -N_{35} = \kappa Q_1 Q_3 X_1
\end{aligned} \tag{28.5}
$$

Consider now

$$
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \boldsymbol{N} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_1^* \\ F_2^* \\ F_3^* \end{bmatrix} \tag{28.6}
$$

If $\boldsymbol{\Phi}_1 = Q_3\boldsymbol{F}$ and $\boldsymbol{\Phi}_2 = Q_1\boldsymbol{F}^*$ through (28.5) and (28.6), we find that

$$
\begin{aligned}
\boldsymbol{u}(\boldsymbol{x},t) &= \square_1 \square_4 \boldsymbol{\Phi}_1 - [(\lambda + \mu)\square_4 - \kappa^2]\boldsymbol{\nabla\nabla} \cdot \boldsymbol{\Phi}_1 - \kappa \square_3 \boldsymbol{\nabla} \times \boldsymbol{\Phi}_2 \\
\boldsymbol{\phi}(\boldsymbol{x},t) &= \square_2 \square_3 \boldsymbol{\Phi}_2 - [(\alpha + \beta)\square_2 - \kappa^2]\boldsymbol{\nabla\nabla} \cdot \boldsymbol{\Phi}_2 - \kappa \square_1 \boldsymbol{\nabla} \times \boldsymbol{\Phi}_1
\end{aligned} \tag{28.7}
$$

where

$$
\begin{aligned}
&\square_1 \equiv (\lambda + 2\mu + \kappa)\nabla^2 - \rho \frac{\partial^2}{\partial t^2} \quad , \quad \square_2 \equiv (\mu + \kappa)\nabla^2 - \rho \frac{\partial^2}{\partial t^2} \\
&\square_2 \equiv (\alpha + \beta + \gamma)\nabla^2 - \rho j \frac{\partial^2}{\partial t^2} - 2\kappa, \quad \square_4 \equiv \gamma\nabla^2 - \rho j \frac{\partial^2}{\partial t^2}, -2\kappa
\end{aligned} \tag{28.8}
$$

From (28.3), (28.4), and (28.7), it follows that $\boldsymbol{\Phi}_1$ and $\boldsymbol{\Phi}_2$ satisfy the following uncoupled equations

$$
\begin{aligned}
\square_1(\square_2 \square_4 + \kappa^2\nabla^2)\boldsymbol{\Phi}_1 &= -\rho \boldsymbol{f} \\
\square_3(\square_2 \square_4 + \kappa^2\nabla^2)\boldsymbol{\Phi}_2 &= -\rho \boldsymbol{\ell}
\end{aligned} \tag{28.9}
$$

If we take $\kappa = 0$ in $(28.9)_1$, we obtain a representation known in the classical theory of elasticity.

In the static case, we set T = 0 in (28.7) and (28.8) and obtain the Galerkin representation for the micropolar elasticity, namely,

$$\underset{\sim}{u}(\underset{\sim}{x}) = (\lambda + 2\mu + \kappa)\nabla^2(\gamma\nabla^2 - 2\kappa)\underset{\sim}{\Phi}_1 - [\gamma(\lambda + \mu)\nabla^2 - \kappa(2\lambda + 2\mu + \kappa)]$$

$$\underset{\sim}{\nabla}\underset{\sim}{\nabla} \cdot \underset{\sim}{\Phi}_1 - \kappa[(\alpha + \beta + \gamma)\nabla^2 - 2\kappa]\underset{\sim}{\nabla} \times \underset{\sim}{\Phi}_2$$

(28.10)

$$\underset{\sim}{\phi}(\underset{\sim}{x}) = (\mu + \kappa)\nabla^2[(\alpha + \beta + \gamma)\nabla^2 - 2\kappa]\underset{\sim}{\Phi}_2 - [(\mu + \kappa)(\alpha + \beta)\nabla^2 - \kappa^2]\underset{\sim}{\nabla}\underset{\sim}{\nabla} \cdot \underset{\sim}{\Phi}_2$$

$$- \kappa(\lambda + 2\mu + \kappa)\nabla^2(\underset{\sim}{\nabla} \times \underset{\sim}{\Phi}_1)$$

where $\underset{\sim}{\Phi}_1$ and $\underset{\sim}{\Phi}_2$ satisfy the equations

$$(\lambda + 2\mu + \kappa)\nabla^4[(\mu + \kappa)\gamma\nabla^2 - \kappa(2\mu + \kappa)]\underset{\sim}{\Phi}_1 = -\rho\underset{\sim}{f}$$

(28.11)

$$[(\alpha + \beta + \gamma)\nabla^2 - 2\kappa][(\mu + \kappa)\gamma\nabla^4 - \kappa(2\mu + \kappa)]\underset{\sim}{\Phi}_2 = -\rho\underset{\sim}{\ell}$$

For $\kappa = 0$. $(28.10)_1$ gives Galerkin's representation of classical elasticity.

We decompose the body force and body couple into irrotational and solenoidal parts as follows:

(28.12)

$$\rho\underset{\sim}{f} = \underset{\sim}{\nabla}\Pi_0 + \underset{\sim}{\nabla} \times \underset{\sim}{\Pi}$$

$$\rho\underset{\sim}{\ell} = \underset{\sim}{\nabla}\Pi_0^* + \underset{\sim}{\nabla} \times \underset{\sim}{\Pi}^*$$

and note that

(28.13)

$$(\Box_2 \Box_4 + \kappa^2\nabla^2)\underset{\sim}{\Phi}_1 = \nabla\Lambda_0 , \quad \Box_1\underset{\sim}{\Phi}_1 = \underset{\sim}{\nabla} \times \underset{\sim}{\Lambda}$$

$$(\Box_2 \Box_4 + \kappa^2\nabla^2)\underset{\sim}{\Phi}_2 = \nabla\Lambda_0^* , \quad \Box_3\underset{\sim}{\Phi}_2 = \underset{\sim}{\nabla} \times \underset{\sim}{\Lambda}^*$$

From (28.7) and (28.9) it now follows that

(28.14) $$\Box_1\Lambda_0 = -\Pi_0 , \quad \Box_3\Lambda_0^* = -\Pi_0^*$$

and we obtain

$$\tilde{u} = \nabla\Lambda_0 + \tilde{\nabla} \times (\Box_4\tilde{\Lambda}) - \kappa\tilde{\nabla} \times (\tilde{\nabla} \times \Lambda^*) \tag{28.15}$$

$$\tilde{\phi} = -\kappa\tilde{\nabla} \times (\tilde{\nabla} \times \tilde{\Lambda}) + \tilde{\nabla} \times (\Box_2\tilde{\Lambda}^*)$$

Thus, if we determine Λ_0, $\tilde{\Lambda}$ and $\tilde{\Lambda}^*$ for a given problem, the displacement and microrotation fields can be calculated from (28.15).

Another useful decomposition can be deduced from (28.11) with $\tilde{f} = \tilde{\ell} = 0$ by selecting $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$ as the sum of three special vector functions. This has the form

$$\tilde{u} = \tilde{A}_1 + \tilde{A}_2 - \frac{2\lambda + 2\mu+\kappa}{4(\lambda+2\mu+\kappa)}\tilde{\nabla}(\tilde{x} \cdot \tilde{A}_1 + A_0) - \frac{(\mu+\kappa)\gamma}{\kappa(2\mu+\kappa)}\tilde{\nabla}\tilde{\nabla} \cdot (\tilde{A}_1 + \tilde{A}_2)$$

$$+ 2\tilde{\nabla} \times \tilde{B}_1 - \frac{\gamma}{2\mu+\kappa}\tilde{\nabla} \times \tilde{B}_3 \tag{28.16}$$

$$\tilde{\phi} = \tilde{\nabla}\tilde{\nabla} \cdot \tilde{B}_2 + \tilde{B}_3 - \frac{(\mu+\kappa)\gamma}{\kappa(2\mu+\kappa)}\tilde{\nabla}\tilde{\nabla} \cdot \tilde{B}_3 + \tilde{\nabla}\tilde{\nabla} \cdot \tilde{B}_1 + \frac{1}{2}\tilde{\nabla} \times \tilde{A}_1 + \frac{\mu+\kappa}{\kappa}\tilde{\nabla} \times \tilde{A}_2$$

where A_0, $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{B}_1$, $\tilde{B}_2$, and $\tilde{B}_3$ satisfy

$$\nabla^2 A_0 = 0, \quad \nabla^2\tilde{A}_1 = \tilde{0}, \quad \left(1 - \frac{(\mu + \kappa)\gamma}{\kappa(2\mu + \kappa)}\nabla^2\right)\tilde{A}_2 = \tilde{0} \tag{28.17}$$

$$\nabla^2\tilde{B}_1 = \tilde{0}, \quad \left(1 - \frac{\alpha + \beta + \gamma}{2\kappa}\nabla^2\right)\tilde{B}_2 = \tilde{0}, \quad \left(1 - \frac{(\mu + \kappa)\gamma}{\kappa(2\mu + \kappa)}\nabla^2\right)\tilde{B}_3 = \tilde{0}$$

These results when $\kappa \to \infty$ reduce to the corresponding decomposition for the indeterminate couple stress theory obtained by Mindlin and Tiersten [1962].

29. MICROPOLAR INFINITE SOLID SUBJECT TO CONCENTRATED FORCE AND CONCENTRATED COUPLE

The problem of an infinite solid subject to a concentrated force $\underset{\sim}{F}$ and a concentrated couple $\underset{\sim}{C}$ at the origin of coordinates $\underset{\sim}{x} = \underset{\sim}{0}$ is of fundamental interest. In classical theory, this problem is known as Kelvin's problem. This problem for the static case was treated by Sandru [1966].

(i) Concentrated Force. Let $\underset{\sim}{F}$ be a concentrated force acting at the origin of the coordinates $\underset{\sim}{x} = \underset{\sim}{0}$. We write

$$\rho \underset{\sim}{f} = \underset{\sim}{F}\delta(\underset{\sim}{x}) \tag{29.1}$$

where $\delta(\underset{\sim}{x})$ is the Dirac delta function.

The solution of $(28.12)_1$ is given by

$$\begin{aligned} \Pi_0(\underset{\sim}{x}) &= -\frac{1}{4\pi}\int_V \rho \underset{\sim}{f}(\underset{\sim}{\xi}) \cdot \underset{\sim}{\nabla}\left(\frac{1}{r}\right) dv(\underset{\sim}{\xi}) \\ \underset{\sim}{\Pi}(\underset{\sim}{x}) &= -\frac{1}{4\pi}\int_V \rho \underset{\sim}{f}(\underset{\sim}{\xi}) \times \underset{\sim}{\nabla}\left(\frac{1}{r}\right) dv(\underset{\sim}{\xi}) \end{aligned} \tag{29.2}$$

where

$$r \equiv [(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2]^{1/2} \tag{29.3}$$

Using (29.1) we find

$$\begin{aligned} \Pi_0 &= -\frac{1}{4\pi}\underset{\sim}{F} \cdot \underset{\sim}{\nabla}\left(\frac{1}{R}\right) \\ \underset{\sim}{\Pi} &= -\frac{1}{4\pi}\underset{\sim}{F} \times \underset{\sim}{\nabla}\left(\frac{1}{R}\right) \end{aligned} \tag{29.4}$$

where

$$R \equiv (x_1^2 + x_2^2 + x_3^2)^{1/2} \tag{29.5}$$

From $(28.13)_{1,2}$ for the static case we get

$$(29.6)\qquad \begin{aligned}(\lambda + 2\mu + \kappa)\nabla^2 \Lambda_0 &= \frac{1}{4\pi} \underset{\sim}{F} \cdot \underset{\sim}{\nabla}(\frac{1}{R})\\ [\gamma(\mu + \kappa)\nabla^4 - \kappa(2\mu + \kappa)]\underset{\sim}{\Lambda} &= \frac{1}{4\pi} \underset{\sim}{F} \times \underset{\sim}{\nabla}(\frac{1}{R})\end{aligned}$$

These equations possess the solutions

$$(29.7)\qquad \begin{aligned}\Lambda_0 &= \frac{1}{8\pi(\lambda+2\mu+\kappa)} \frac{\underset{\sim}{F} \cdot \underset{\sim}{x}}{R}\\ \underset{\sim}{\Lambda} &= \frac{1}{8\pi\kappa(2\mu+\kappa)} \underset{\sim}{\nabla} \times (\underset{\sim}{F}R) + \frac{\ell^2}{4\pi\kappa(2\mu+\kappa)} \underset{\sim}{\nabla} \times [\frac{\underset{\sim}{F}}{R} (1 - e^{-R/\ell})]\end{aligned}$$

where

$$(29.8)\qquad \ell^2 \equiv \frac{\gamma(\mu + \kappa)}{(2\mu + \kappa)\kappa}$$

With $\underset{\sim}{\Lambda}^* = \underset{\sim}{0}$, (28.15) gives for the static case

$$(29.9)\qquad \begin{aligned}\underset{\sim}{u} &= \frac{5\lambda+6\mu+3\kappa}{8\pi(2\mu+\kappa)(\lambda+2\mu+\kappa)} \frac{\underset{\sim}{F}}{R} + \frac{2\lambda+2\mu+\kappa}{8\pi(2\mu+\kappa)(\lambda+2\mu+\kappa)} \frac{\underset{\sim}{F} \cdot \underset{\sim}{x}}{R^3} \underset{\sim}{x}\\ &\quad + \frac{\gamma}{4\pi(2\mu+\kappa)^2} \underset{\sim}{\nabla} \times \{\underset{\sim}{\nabla} \times [\frac{\underset{\sim}{F}}{R} (e^{-R/\ell}-1)]\}\\ \underset{\sim}{\phi} &= \frac{1}{4\pi(2\mu+\kappa)} \underset{\sim}{\nabla} \times [\frac{\underset{\sim}{F}}{R} (1 - e^{-R/\ell})]\end{aligned}$$

For $\kappa = \gamma = 0$, $(29.9)_1$ reduces to the well-known solution of Kelvin's problem, Love, [1944, p. 185]. The solution of this problem for the indeterminate couple stress theory is obtained by letting $\kappa \to \infty$. This gives the result obtained by Mindlin and Tiersten [1962].

(ii) Concentrated Couple. Let $\underset{\sim}{M}$ be a concentrated couple acting at $\underset{\sim}{x} = \underset{\sim}{0}$. We write

$$\rho\underset{\sim}{\ell} = \underset{\sim}{M}\delta(\underset{\sim}{x})$$

In this case, we express the solution of (27.12) in the same way as in (28.2). Using $(28.13)_{2,3}$ for the static case we have

$$(29.10)\qquad \begin{aligned} &[(\alpha + \beta + \gamma)\nabla^2 - 2\kappa]\Lambda_0^* = -\Pi_0^* \\ &[\gamma(\mu + \kappa)\nabla^4 - \kappa(2\mu + \kappa)\nabla^2]\underset{\sim}{\Lambda}^* = -\underset{\sim}{\Pi}^* \end{aligned}$$

where

$$(29.11)\qquad \Pi_0^* = -\frac{1}{4\pi}\underset{\sim}{M}\cdot\underset{\sim}{\nabla}\left(\frac{1}{R}\right), \quad \underset{\sim}{\Pi}^* = -\frac{1}{4\pi}\underset{\sim}{M}\times\underset{\sim}{\nabla}\left(\frac{1}{R}\right)$$

Equations (29.10) have the solutions

$$(29.12)\qquad \begin{aligned} &\Lambda_0^* = \frac{1}{8\pi\kappa}\underset{\sim}{M}\cdot\underset{\sim}{\nabla}\left[\frac{1}{R}\left(e^{-R/h} - 1\right)\right] \\ &\underset{\sim}{\Lambda}^* = \frac{1}{8\pi(2\mu+\kappa)\kappa}\underset{\sim}{\nabla}\times(\underset{\sim}{M}R) + \frac{\ell^2}{4\pi(2\mu+\kappa)\kappa}\underset{\sim}{\nabla}\times\left[\frac{\underset{\sim}{M}}{R}\left(1 - e^{-R/\ell}\right)\right] \end{aligned}$$

where

$$(29.13)\qquad h^2 \equiv \frac{\alpha + \beta + \gamma}{2\kappa}$$

Substituting these results into (28.15) for the static case, and with $\Lambda_0 = 0$ and $\underset{\sim}{\Lambda} = \underset{\sim}{0}$, we obtain

$$(29.14)\qquad \begin{aligned} &\underset{\sim}{u} = \frac{1}{4\pi(2\mu+\kappa)}\underset{\sim}{\nabla}\times\left[\frac{\underset{\sim}{M}}{R}\left(1 - e^{-R/\ell}\right)\right] \\ &\underset{\sim}{\phi} = \frac{1}{8\pi\kappa}\underset{\sim}{\nabla}\left\{\underset{\sim}{M}\cdot\underset{\sim}{\nabla}\left[\frac{1}{R}\left(e^{-R/h}-1\right)\right]\right\} + \frac{\mu+\kappa}{4\pi(2\mu+\kappa)\kappa}\underset{\sim}{\nabla}\times\left\{\underset{\sim}{\nabla}\times\left[\frac{\underset{\sim}{M}}{R}\left(1 - e^{-R/\ell}\right)\right]\right\} \end{aligned}$$

These results may be used in (20.23) and (20.24) to obtain the stress and couple stress fields.

It is worth noting here that the concentrated couple is a fundamental problem which is not deduced as the limiting solution of two equal parallel forces directed in opposite sense. We note that in micropolar elasticity, the concept of body couple is totally independent of the force and it can exist even when the body force is absent. For micropolar elasticity, therefore, force and moment singularities will have totally different natures. This is then expected to affect uniqueness theorems for infinite and for the finite regions considerably.

SUMMARY

The theory of micropolar elasticity presented here should provide an adequate background for pursuing analytical work and starting a badly needed experimental program. We believe the theory is *well-posed* with field equations, boundary and initial conditions. Certain wide classes of uniqueness theorems have been proved though not presented here (cf. Eringen [1966a]). The important implications of the theory are brought to the surface especially in connection with the problems in the field of wave propagation. Existence of additional waves over those existing in classical elasticity should be attractive to workers in the field of experimental wave propagation. The linear theory is simple enough to lend itself to the solution of non-trivial boundary and initial value problems. With the additional internal degrees of freedom provided by the microrotation and spin inertia, it incorporates the problems involving concentrated body and surface couples into the *fundamental* singularities of the field.

The field is rather new and not even partly explored. Experimental works are badly needed. Nevertheless, it seems to us that the logical foundation of the theory is solid and promising for the understanding of the mechanics of micropolar solids.

POSSIBLE DIRECTIONS FOR FUTURE RESEARCH

The theory of micropolar elasticity presented here is one of the simplest extensions of the classical theory of elasticity for the treatment of materials with microstructure. Extensions of this theory to micropolar fluids and viscoelasticity exist (cf. Eringen [1964], [1966b], [1967]). A nonlinear theory is also contained in the more general theory of microelasticity given by Eringen and Suhubi [1964a,b]. The theory of micromorphic materials, of which the theory of microelasticity is a representative field for the oriented solids, possesses promise for entering the granular and molecular world of materials from the continuum side. It should not be surprising if such a theory would be in wide use for the full description of material properties of composites, granular and fiberous solids. The connection of these theories to continuum dislocation theory has already been exhibited. The intimate ties between the plasticity theory and continuum dislocations has been recognized for some time by some research workers in this field (cf., Kondo [1962], [1963], Kröner [1963], Bilby [1960], Bilby, Gardner and Stroh [1967]). Existing theories of micromechanics, multipolar theories, and the continuum theory of dislocations have not been amalgamated into a unified structure as yet, although some attempts exist in this direction. Presently, serious efforts are being made to bring some order to the world of the microcontinuum. The field of micromorphic materials needs and deserves attention from both theoretical and experimental workers. Especially, the need for rational experiments is felt badly. The theory of micropolar elasticity is certainly ready for such a test. The theory of micromorphic materials with its logical structure and wide possible applications opens new and promised rich lands for future workers.

ACKNOWLEDGEMENT

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LIST OF SYMBOLS

a_k	Acceleration vector
$a_{k\ell m}$	Third-order microdeformation rate tensor
$b_{k\ell}$	Second-order microdeformation rate tensor
c	Wave speed
c_k	Micropolar elastic wave speeds
$c_{k\ell}$, C_{KL}	Spatial and material deformation tensors
d	Differential operator as in dx
da	Area element
dv, dV	Spatial and material volume elements
$d_{k\ell}$	Deformation rate tensor
$\frac{D}{Dt} \equiv \dot{(\ )}$	Material derivative operator
$e_{k\ell}$, E_{KL}	Spatial and material strain tensors
f_k	Body force per unit mass
h	Heat source per unit mass
$i_{k\ell}$, I_{KL}	Spatial and material microinertia tensors
j, J	Jacobians
ℓ_k	Body couple per unit mass
$m_{k\ell}$	Couple stress tensor
$m_{(\underset{\sim}{n})}$, $m_{(\underset{\sim}{n})k}$	Surface couple
n_k, $\underset{\sim}{n}$	Exterior normal vector to a surface
q_k, $\underset{\sim}{q}$	Heat flux vector directed outward of the surface
r_k, R_K	Spatial and material macrorotation vectors
$r_{k\ell}$, R_{KL}	Spatial and material macrorotation tensors
s, S	Spatial and material surfaces
t	Time

$t_{k\ell}$	Stress tensor
$\underset{\sim}{t}_{(\underset{\sim}{n})}$, $t_{(\underset{\sim}{n})k}$	Surface traction
u_k, U_K	Spatial and material displacement vectors
v, V	Spatial and material volumes
$\underset{\sim}{v}$, v_k	Velocity vector
w_k	Vorticity vector
$w_{k\ell}$	Vorticity tensor
$\underset{\sim}{x}$, x_k	Spatial rectangular coordinates
$\underset{\sim}{X}$, X_K	Material rectangular coordinates
α	Micropolar elastic constant
$\underset{\sim}{\alpha}$, α_k	Microacceleration vector
β	Micropolar elastic constant
γ	Micropolar elastic constant
$\gamma_{k\ell m}$, Γ_{KLM}	Third-order microstrain tensors
$\delta_{k\ell}$, δ_{KL}	Kronecker deltas (=1 when indices take the same number, zero otherwise)
ε	Internal energy density per unit mass
$\varepsilon_{k\ell}$, E_{KL}	Second-order microstrain tensors
$\varepsilon_{k\ell m}$	Permutation symbol ($\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = -\varepsilon_{213} = -\varepsilon_{132} = -\varepsilon_{321} = 1$ and zero otherwise)
η	Entropy density per unit mass
θ	Absolute temperature
κ	Micropolar elastic constant
λ, μ	Elastic constants
$\underset{\sim}{\nu}$, ν_k	Gyration vector
$\nu_{k\ell}$	Gyration tensor
$\underset{\sim}{\xi}$, ξ_k	Spatial relative position vector
$\underset{\sim}{\Xi}$, Ξ_K	Material relative position vector

ρ	Mass density
$\underset{\sim}{\sigma}$, σ_k	Intrinsic spin vector
$\underset{\sim}{\phi}$, ϕ_k	Spatial microrotation vector
$\phi_{k\ell}$, Φ_{KL}	Microdisplacement tensors
$\underset{\sim}{\Phi}$, Φ_K	Material microrotation vector
$\underset{\sim}{\chi}_k$, χ_{Kk}	Spatial microdisplacement vectors
$\underset{\sim}{\chi}_K$, χ_{kK}	Material microdisplacement vectors
ψ	Helmholtz free energy
$\psi_{k\ell}$, Ψ_{KL}	Microdeformation tensors
I, II, III	Invariants of tensors
$\underset{\sim}{\nabla}$	Gradient operator
∇^2	Laplacian operator
$\Box$	Wave operator

Cartesian tensor notation is used where indices take values 1,2,3. Repeated indices indicate summation over the range 1,2,3 unless otherwise stated. Indices following a comma indicate partial differentiation, e.g., $x_{k,K} \equiv \partial x_k/\partial X_K$. A superposed dot indicates time rate with material point fixed, e.g., $\dot{\underset{\sim}{x}} \equiv \frac{\partial \underset{\sim}{x}}{\partial t}\Big|_{\underset{\sim}{X}}$.

Indices enclosed within parenthesis and brackets indicate symmetric and antisymmetric parts, e.g.,

$$\varepsilon_{(k\ell)} \equiv \frac{1}{2}(\varepsilon_{k\ell} + \varepsilon_{\ell k}) \ , \ \varepsilon_{[k\ell]} \equiv \frac{1}{2}(\varepsilon_{k\ell} - \varepsilon_{\ell k})$$

To convert into expanded engineering notation use

$$x_1 = x \ , \ x_2 = y \ , \ x_3 = z$$

$$u_1 = u_x, \ u_2 = u_y, \ u_3 = u_z$$

$$t_{11} = t_{xx} = \sigma_x, \ t_{12} = t_{xy} = \tau_{xy}, \ \ldots$$

where σ_x, τ_{xy},... are the conventional stress components sometimes used in engineering literature.

FIGURE CAPTIONS

Fig. 1.1 Mass density versus volume

Fig. 2.1 Material and spatial coordinates

Fig. 2.2 Deformation of a microvolume

Fig. 3.1 Displacement vectors

Fig. 4.1 Microrotation

Fig. 5.1 Deformation of a rectangular parallelepiped

Fig. 5.2 Average rotation

Fig. 5.3 Macrodeformation and microdeformation

Fig. 5.4 Microdeformation with $\underset{\sim}{\Xi}$ = fixed

Fig. 5.5 Microdeformation with X = fixed (mini-deformation)

Fig. 9.1 Uniform microdelatation

Fig. 9.2 Uniaxial strain

Fig. 14.1 Macrovolume element with forces acting on microelements

Fig. 14.2 Macroelement with equipollent force and couple

Fig. 14.3 Extrinsic surface loads on a macrosurface Δa

Fig. 14.4 Equipollent force and couple on Δa

Fig. 14.5 Surface and body loads

Fig. 15.1 Micromass elements containing micromass elements

Fig. 16.1 Surface loads

Fig. 16.2 A tetrahedron with surface loads

Fig. 16.3 Stress tensor

Fig. 16.4 Couple stress tensor

Fig. 24.1 Longitudinal displacement and microrotation waves

Fig. 24.2 Coupled transverse vector waves propagating with speeds v_3 and v_4

Fig. 24.3 Sketch of v_2^2 versus ω

Fig. 24.4 Sketch of v_3^2 and v_4^2 versus ω

Fig. 25.1 Reflection of a longitudinal displacement wave

Fig. 27.1 Cylindrical hole in tension field

Fig. 27.2 Stress concentration factors for $\frac{b}{c} = 0.20$, $\nu = 0$

Fig. 27.3 Stress concentration factors for $\frac{b}{c} = 0.20$, $\nu = 0.5$

Fig. 27.4 Stress concentration factors for $\frac{b}{c} = 0.10$, $\nu = 0$

Fig. 27.5 Stress concentration factors for $\frac{b}{c} = 0.10$, $\nu = 0.5$

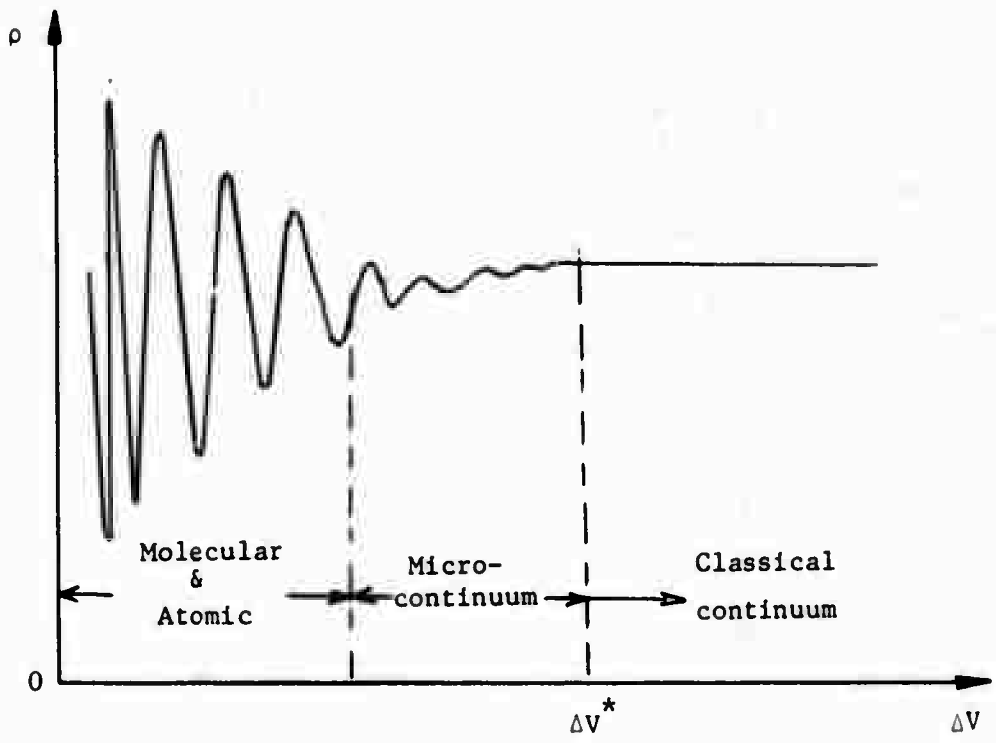

FIG. 1.1. MASS DENSITY VERSUS VOLUME

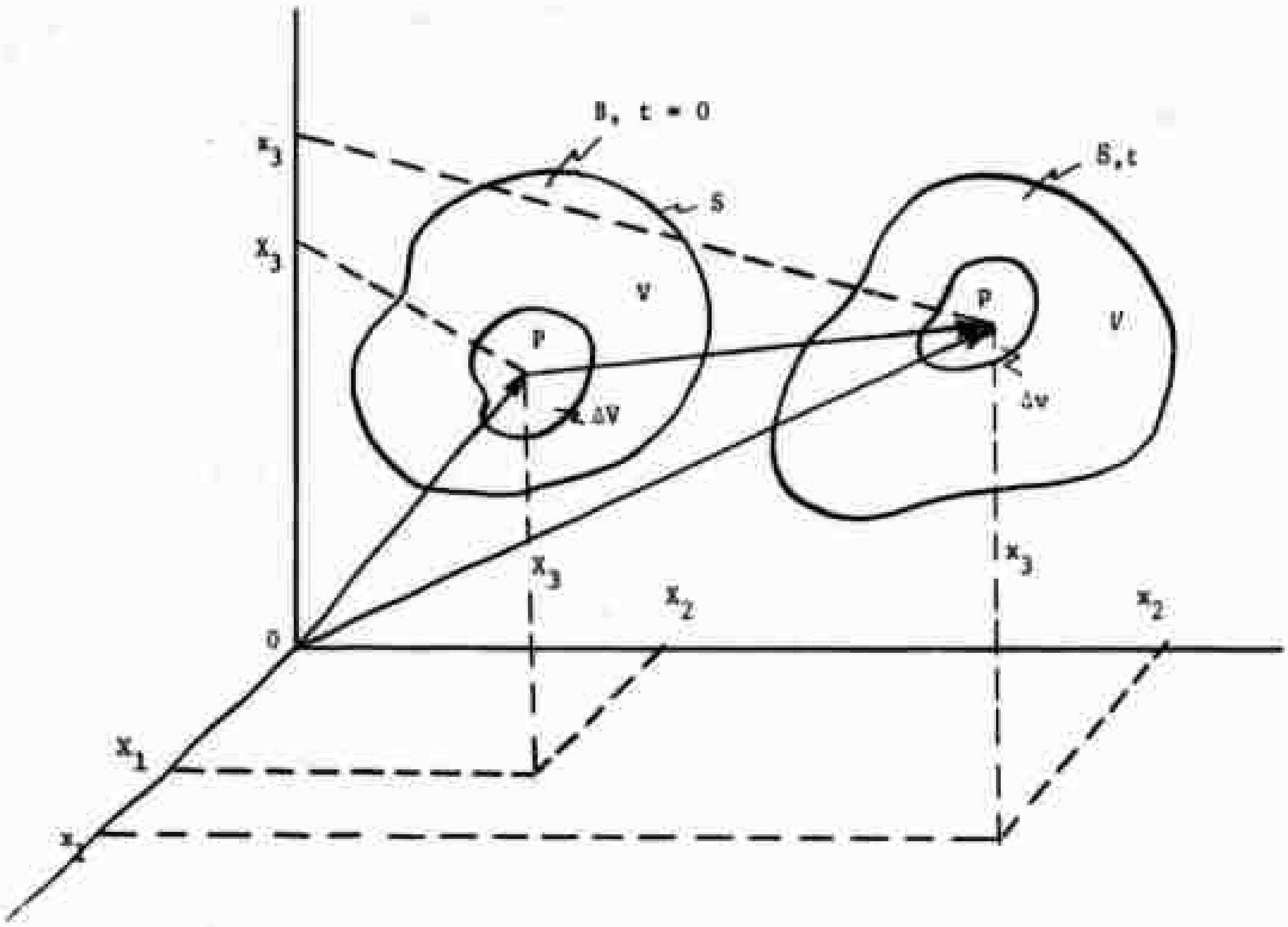

FIG. 2.1. MATERIAL AND SPATIAL COORDINATES

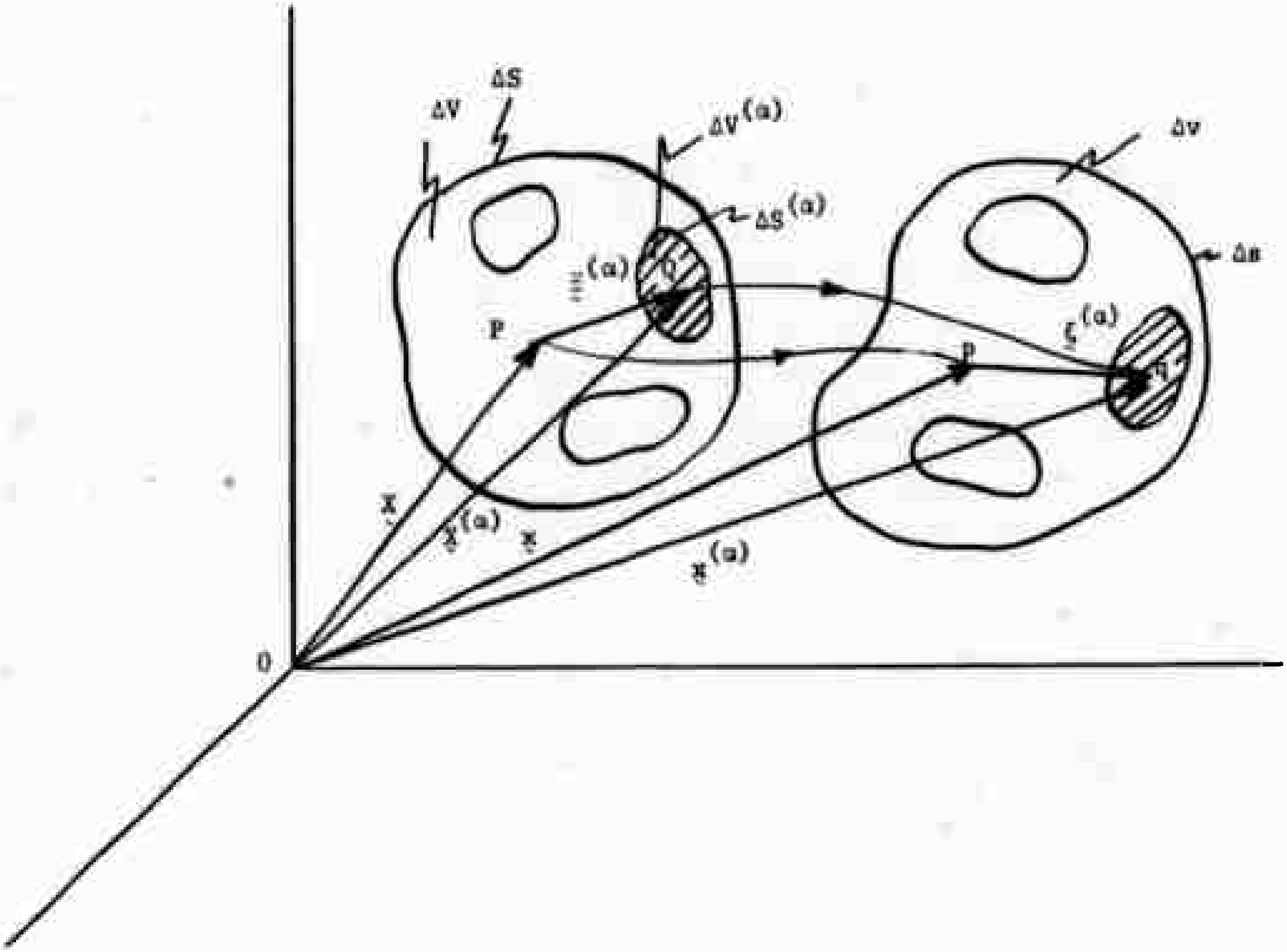

FIG. 2.2. DEFORMATION OF A MICROVOLUME

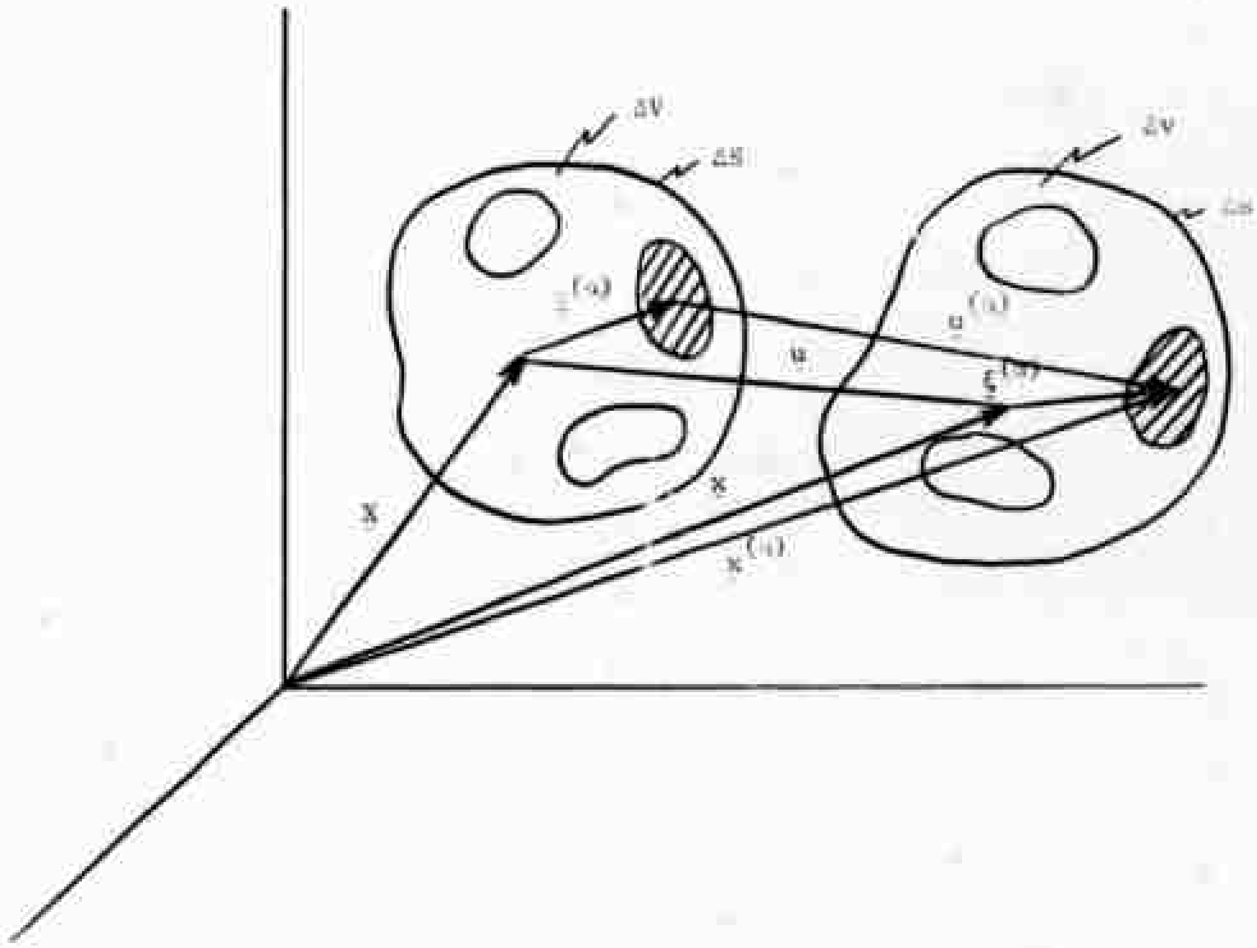

FIG. 3.1. DISPLACEMENT VECTORS

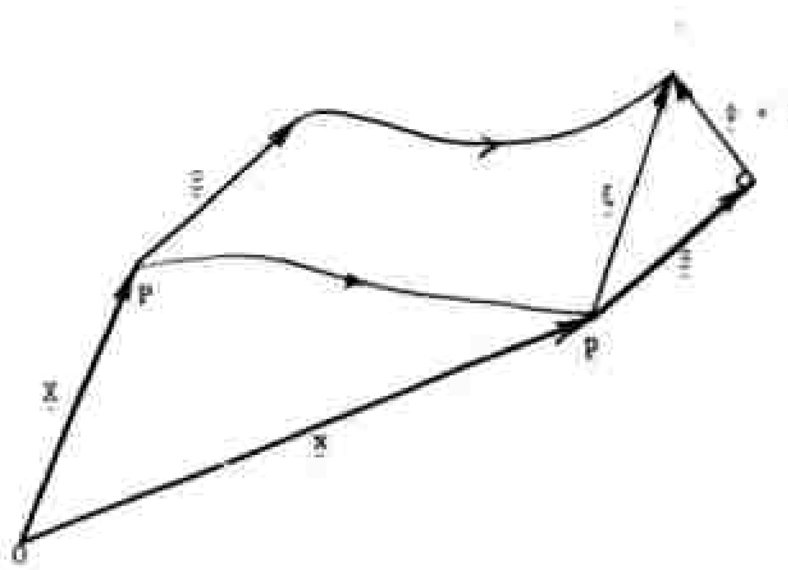

FIG. 4.1. MICROROTATION

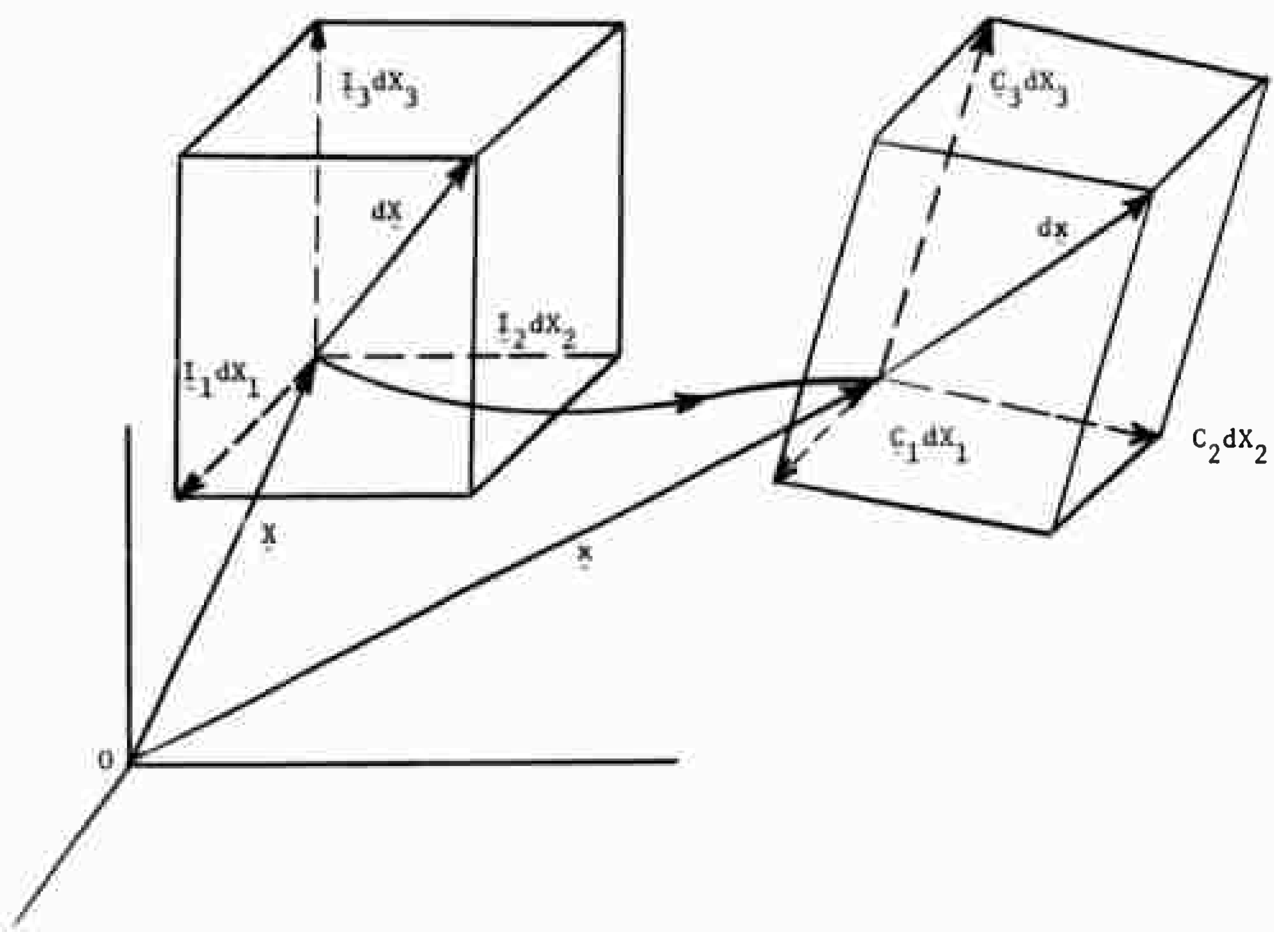

FIG. 5.1. DEFORMATION OF A RECTANGULAR PARALLELEPIPED

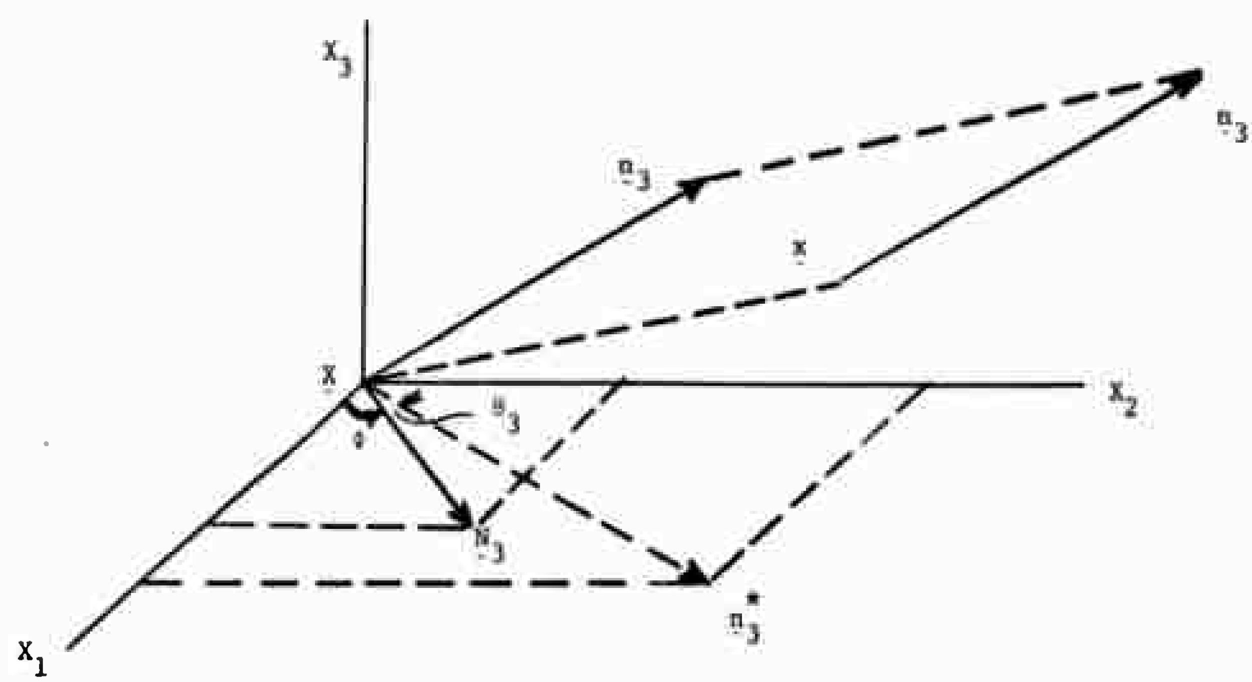

FIG. 5.2. AVERAGE ROTATION

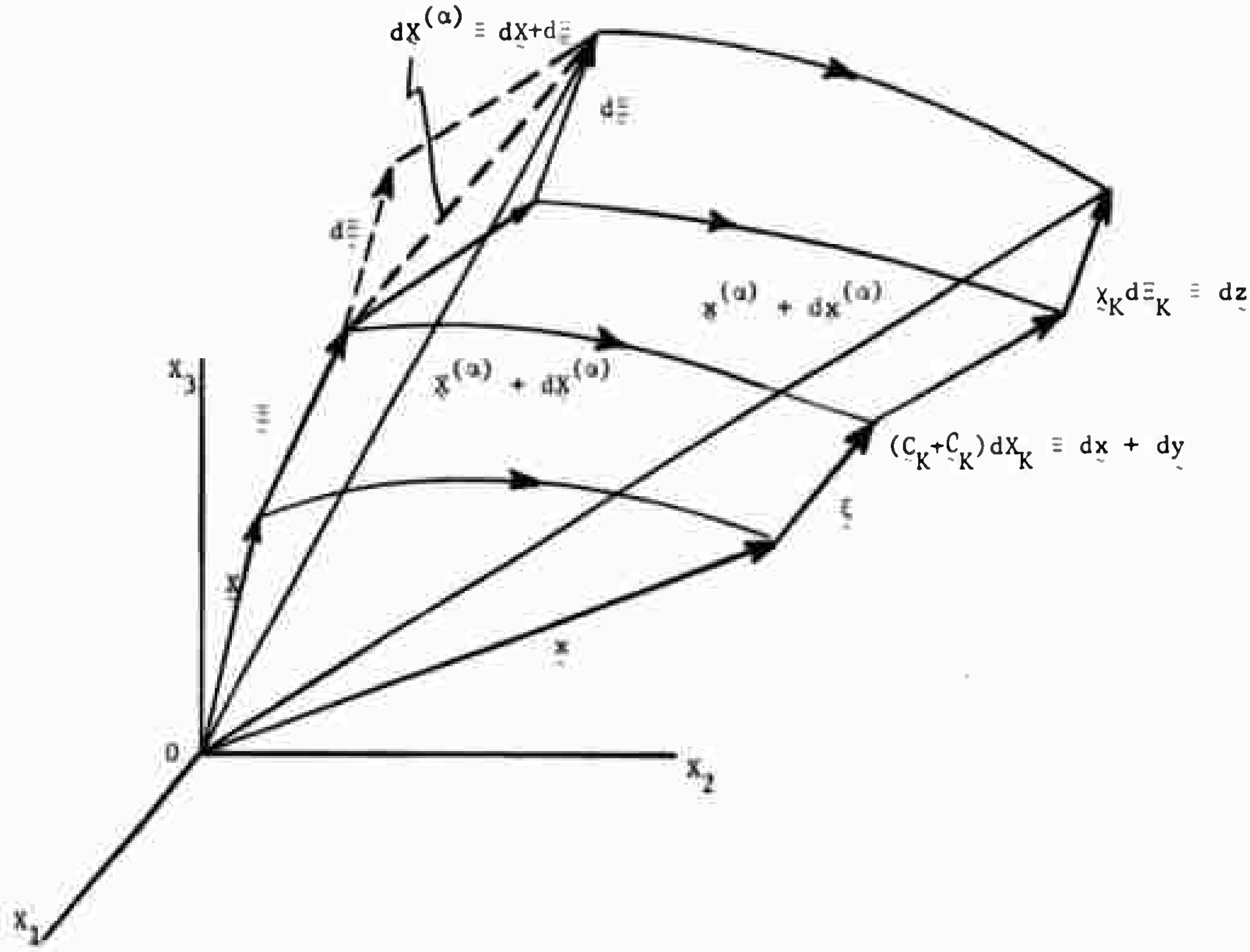

FIG. 5.3. MACRODEFORMATION AND MICRODEFORMATION

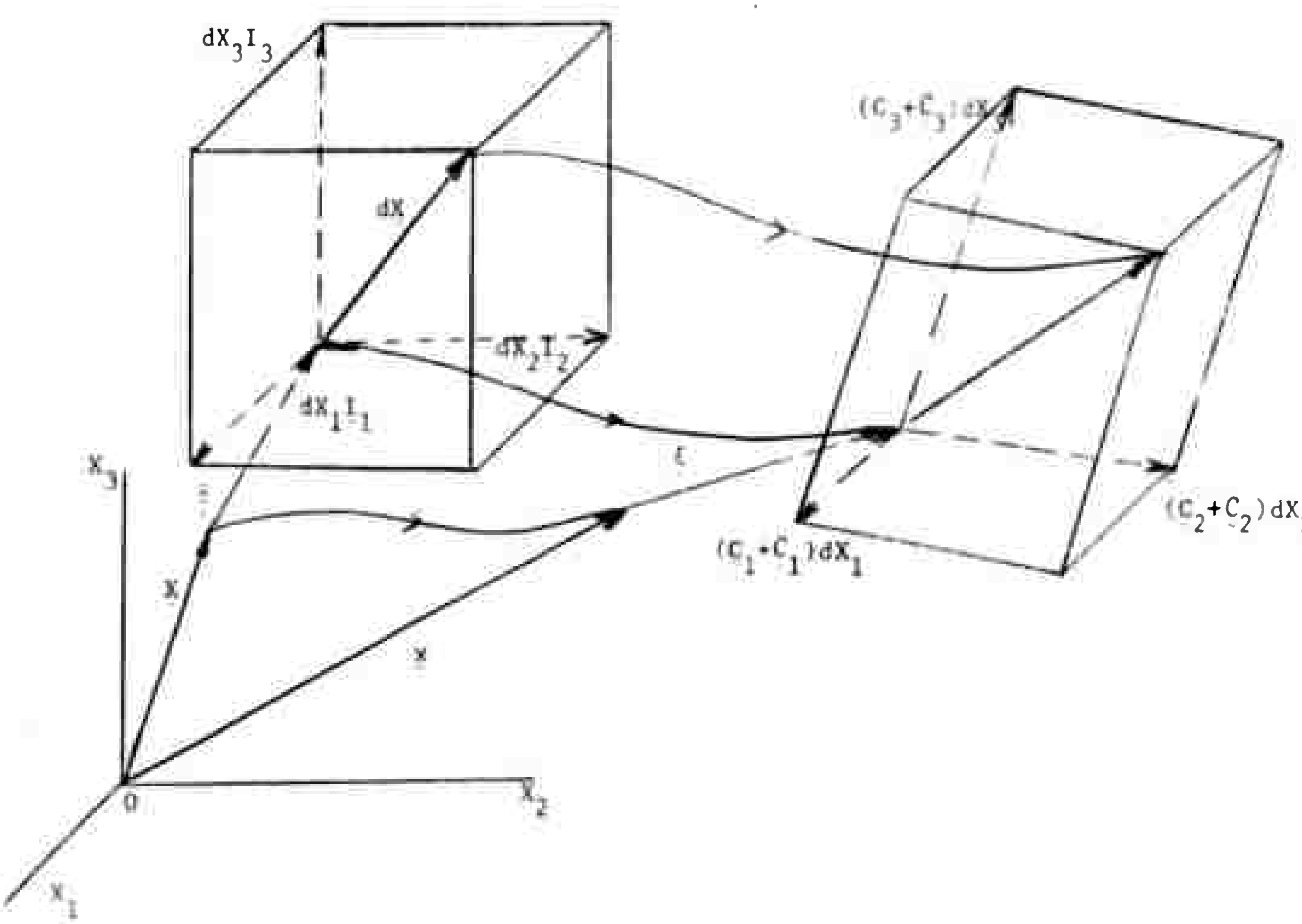

FIG. 5.4 MICRODEFORMATION WITH Ξ - HELD CONSTANT

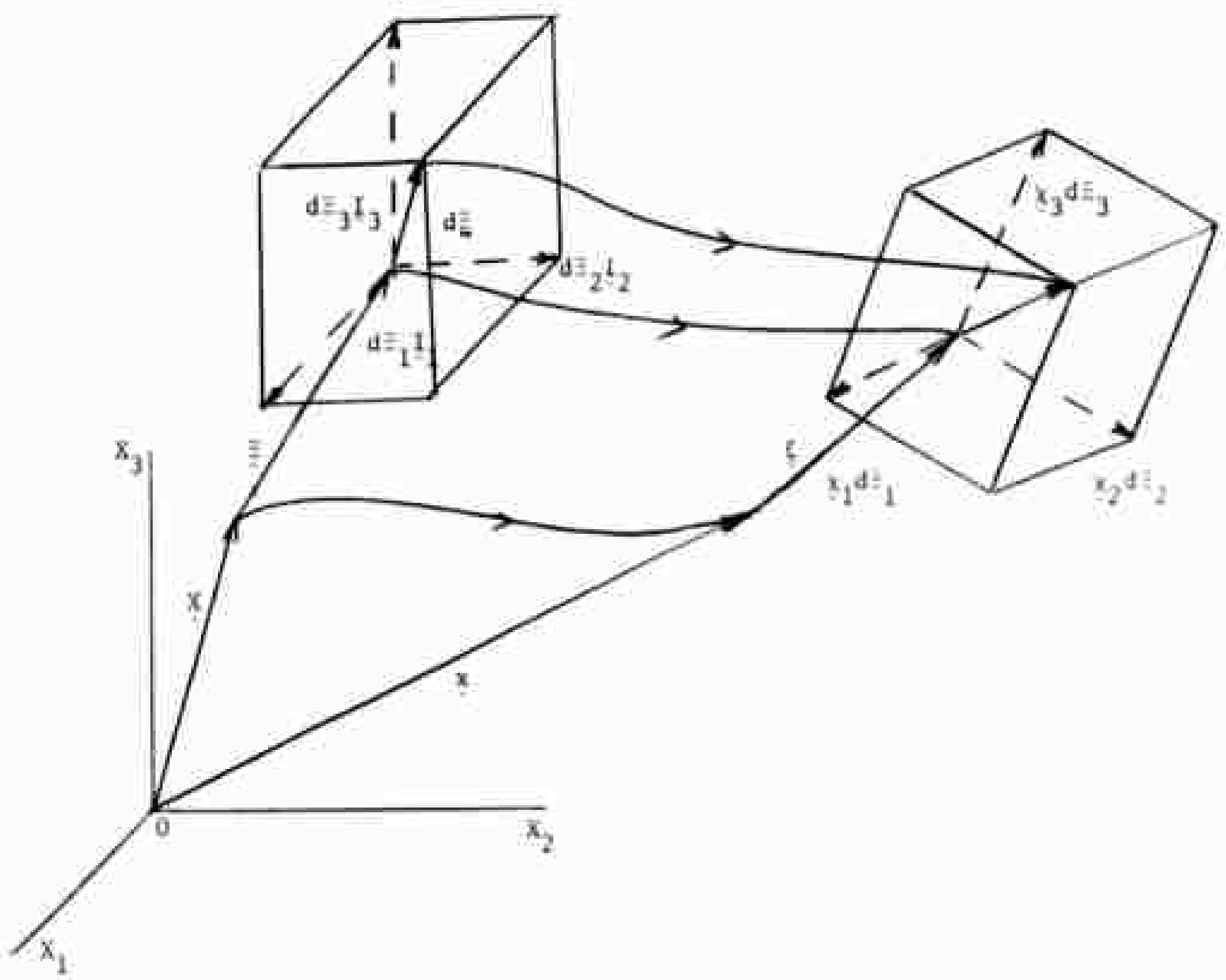

FIG. 5.5. MICRODEFORMATION WITH $\underset{\sim}{X}$ = HELD CONSTANT (MINI-DEFORMATION)

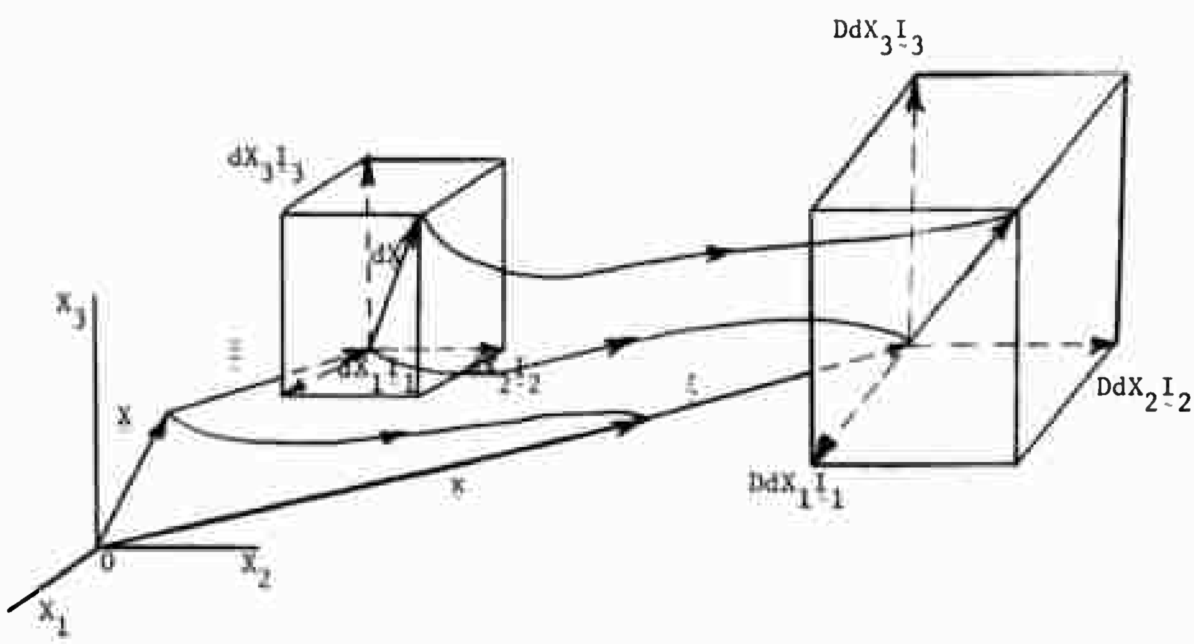

FIG. 9.1. UNIFORM MICRODELATATION

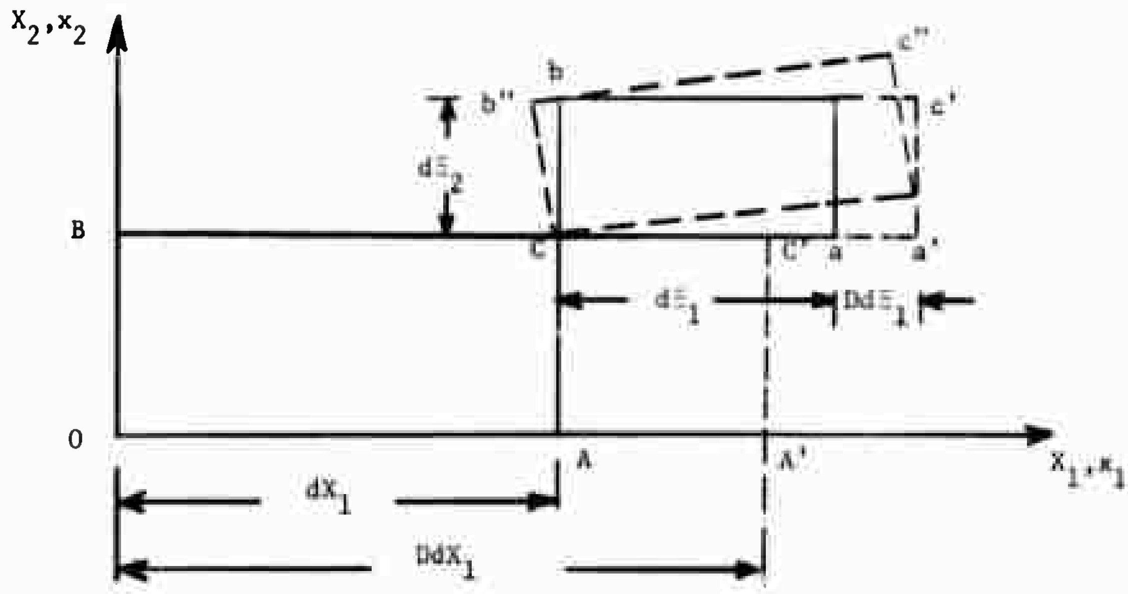

FIG. 9.2. UNIAXIAL STRAIN

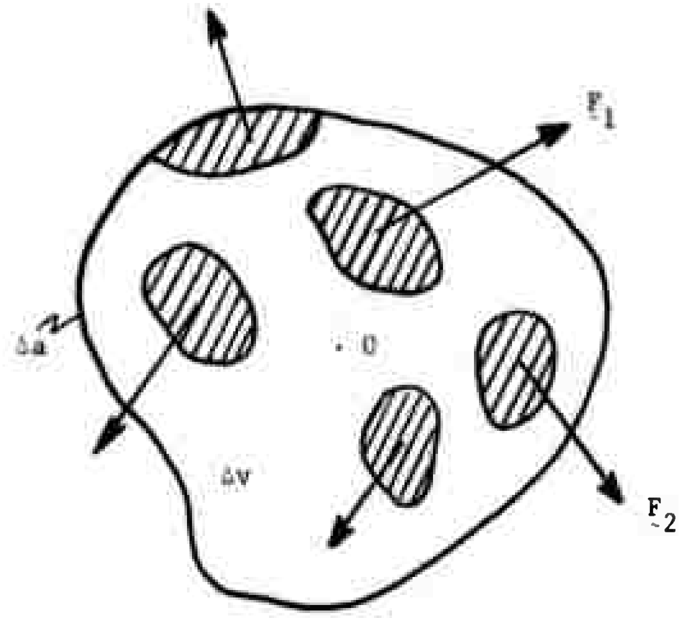

FIG. 14.1. MACROVOLUME ELEMENT WITH FORCES ACTING ON MICROELEMENTS

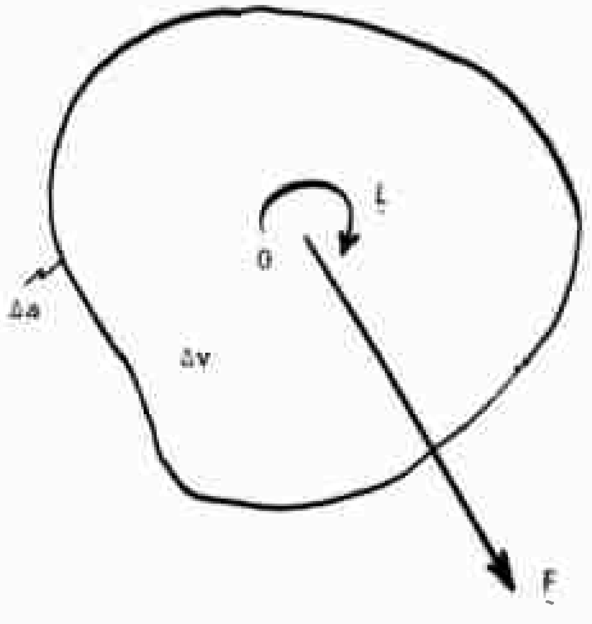

FIG. 14.2. MACROELEMENT WITH EQUIPOLLENT
FORCE AND COUPLE

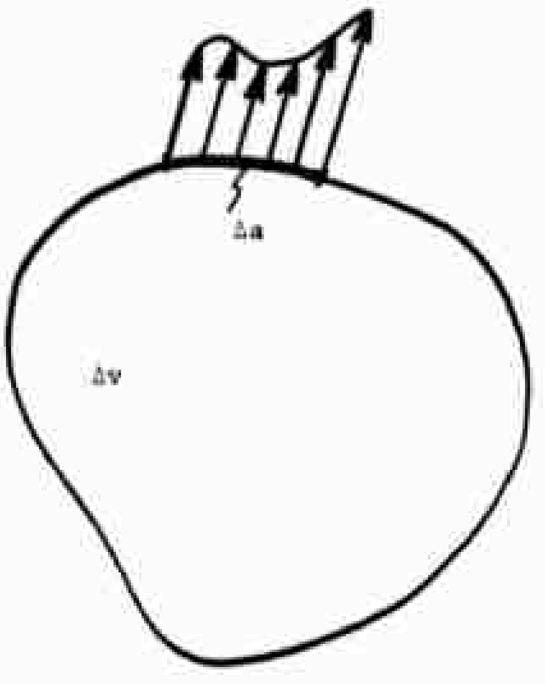

FIG. 14.3. EXTRINSIC SURFACE LOADS ON
A MACROSURFACE Δa

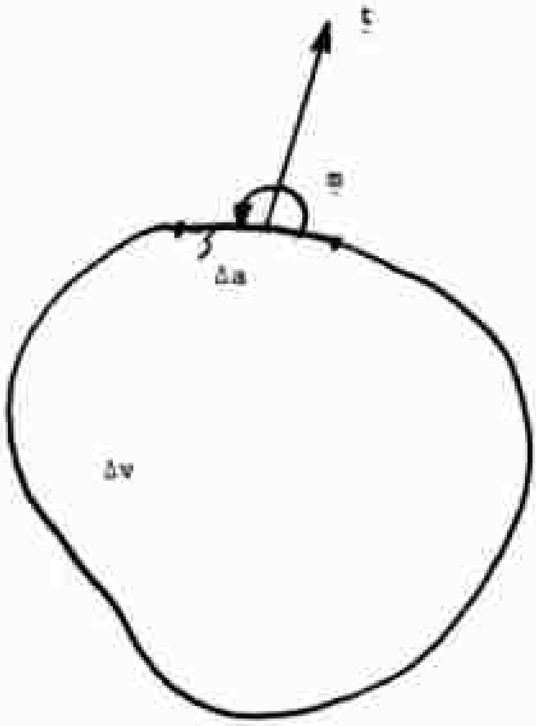

FIG. 14.4. EQUIPOLLENT FORCE AND COUPLE ON Δa

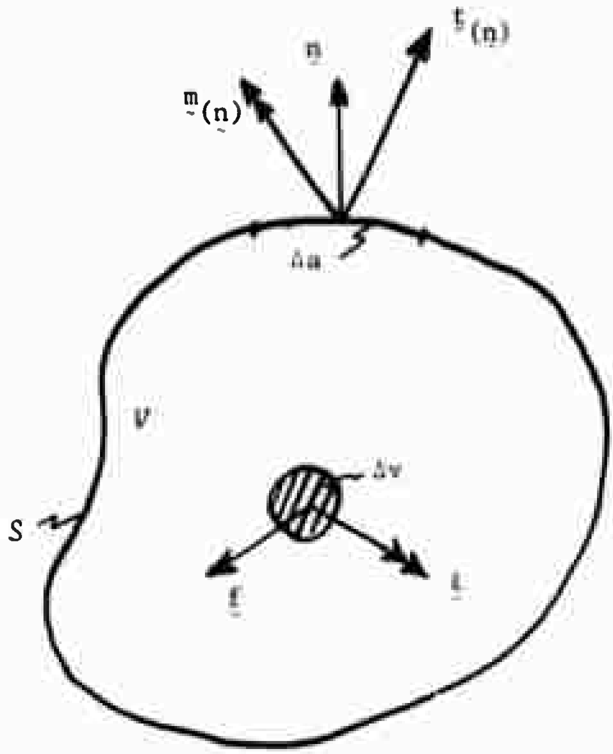

FIG. 14.5. SURFACE AND BODY LOADS

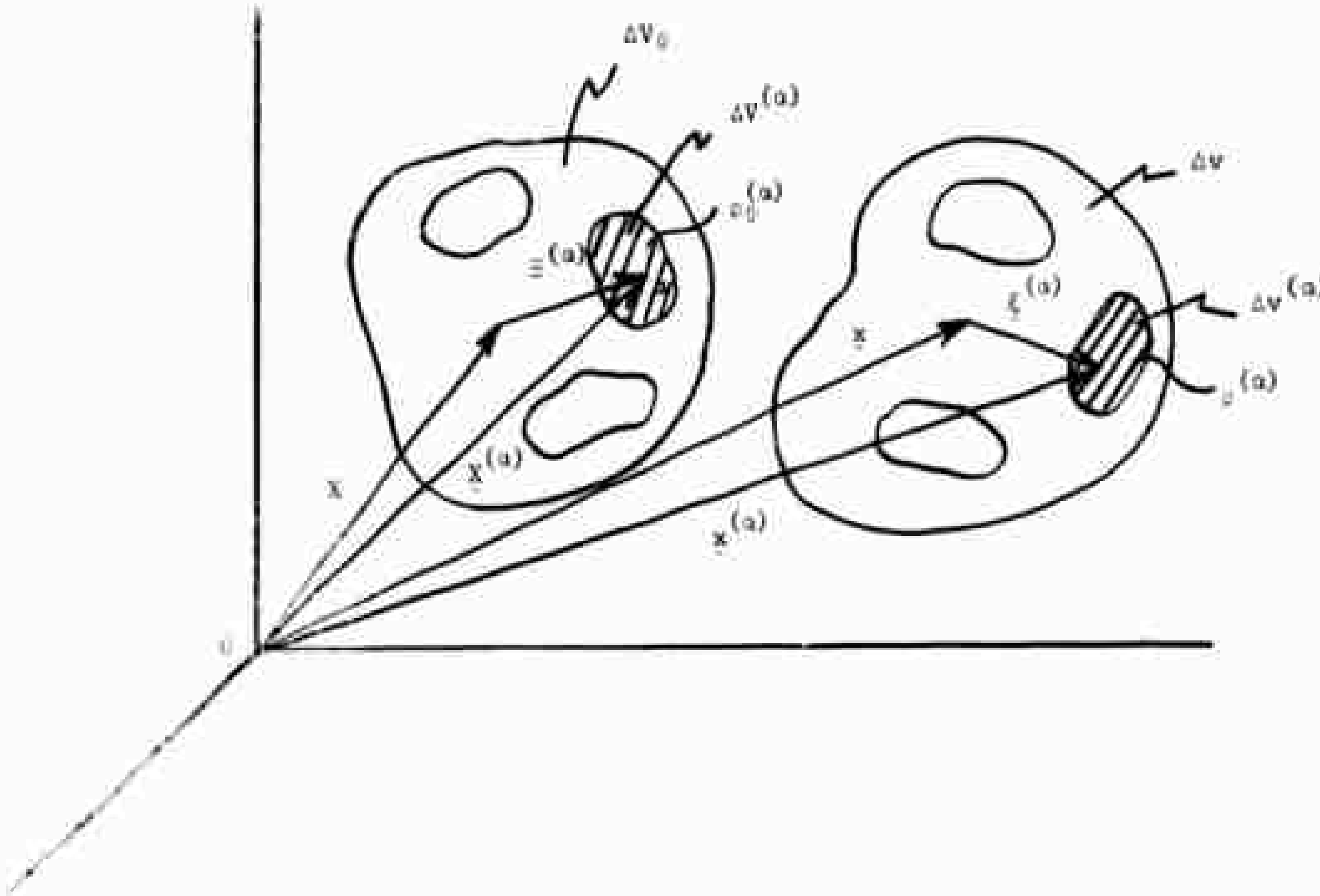

FIG. 15.1. MACROMASS ELEMENTS CONTAINING MICROMASS ELEMENTS

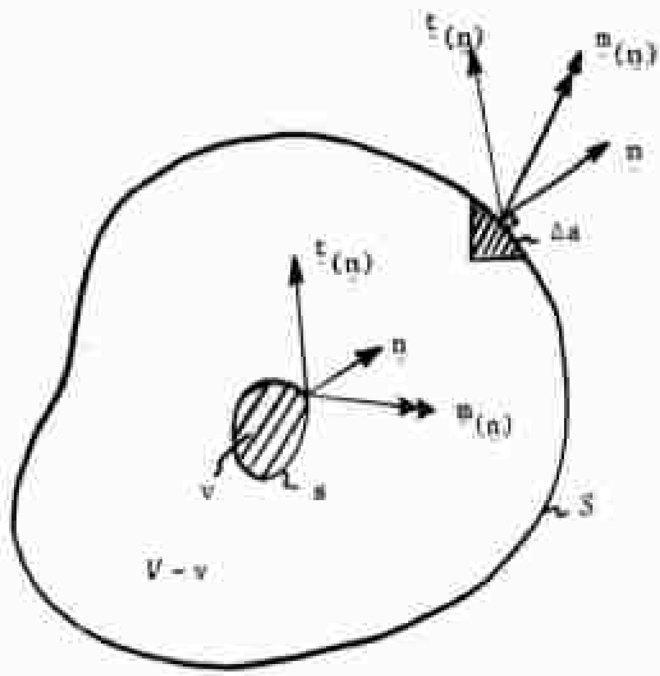

FIG. 16.1. SURFACE LOADS

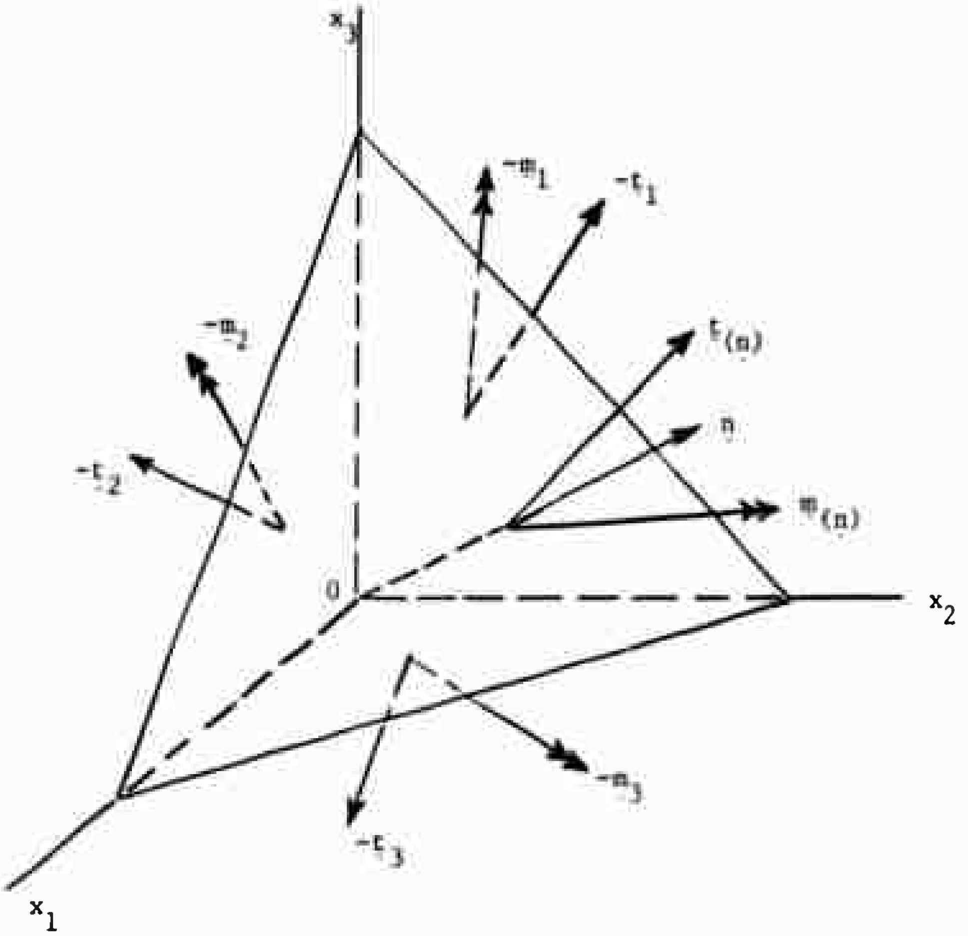

FIG. 16.2. A TETRAHEDRON WITH SURFACE LOADS

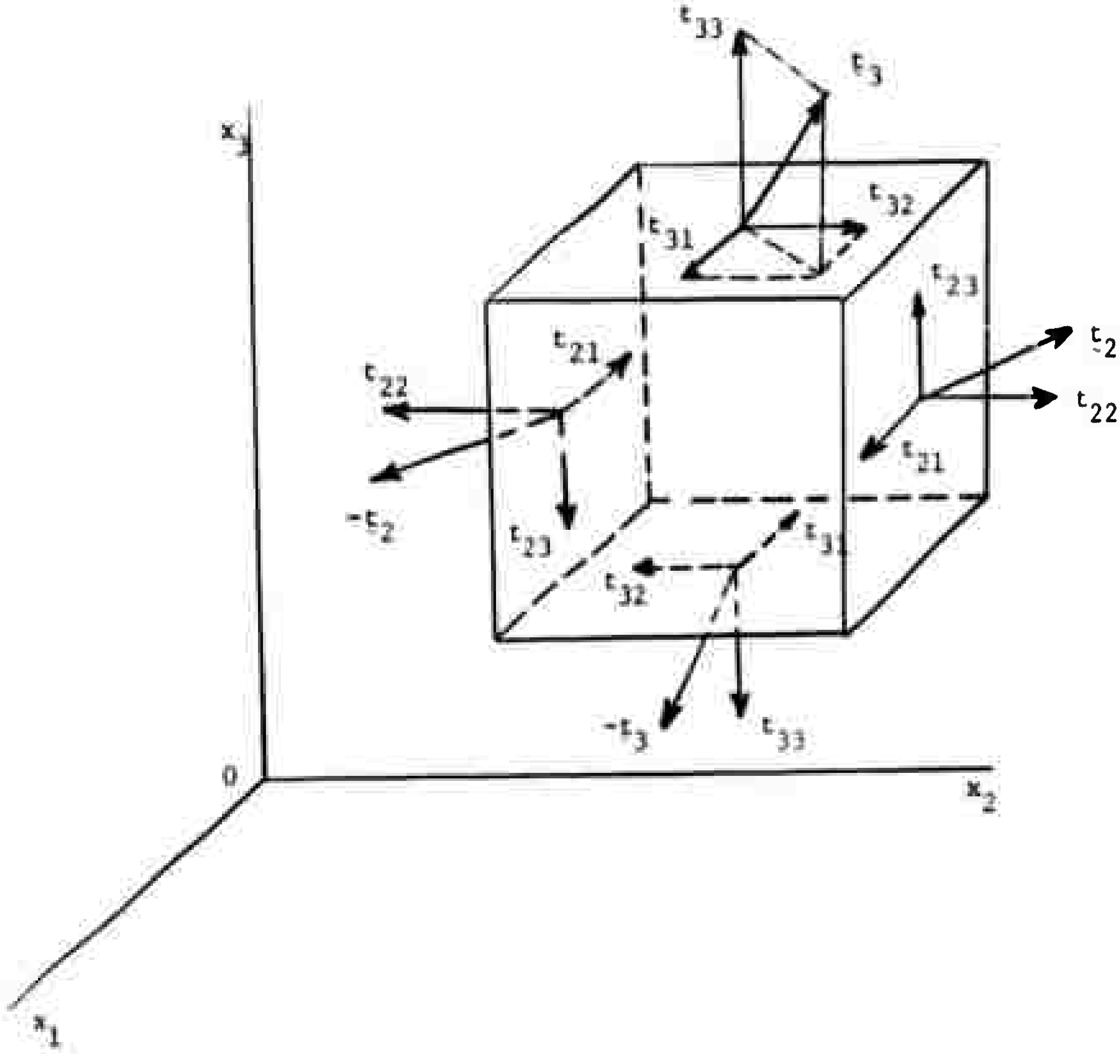

FIG. 16.3. STRESS TENSOR

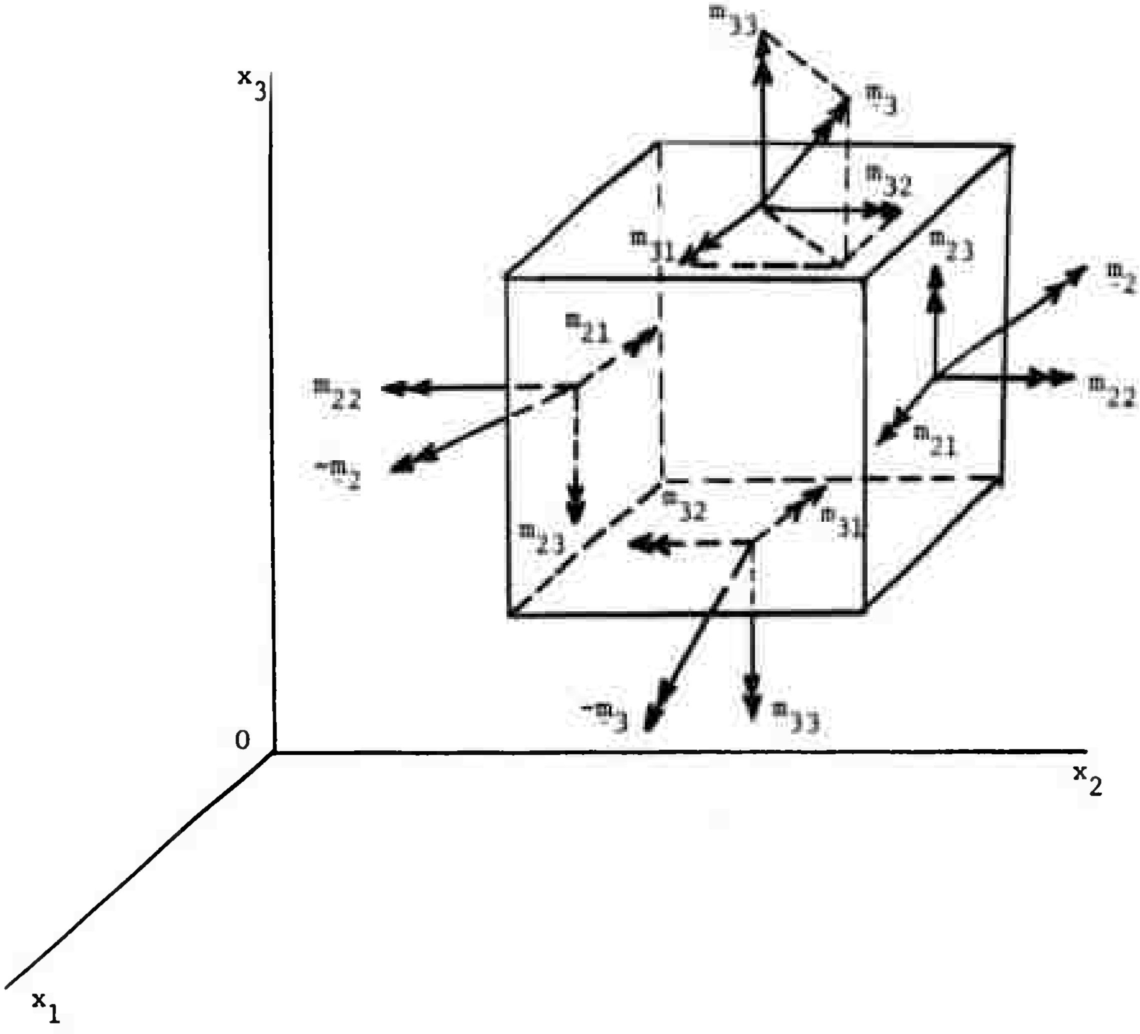

FIG. 16.4. COUPLE STRESS TENSOR

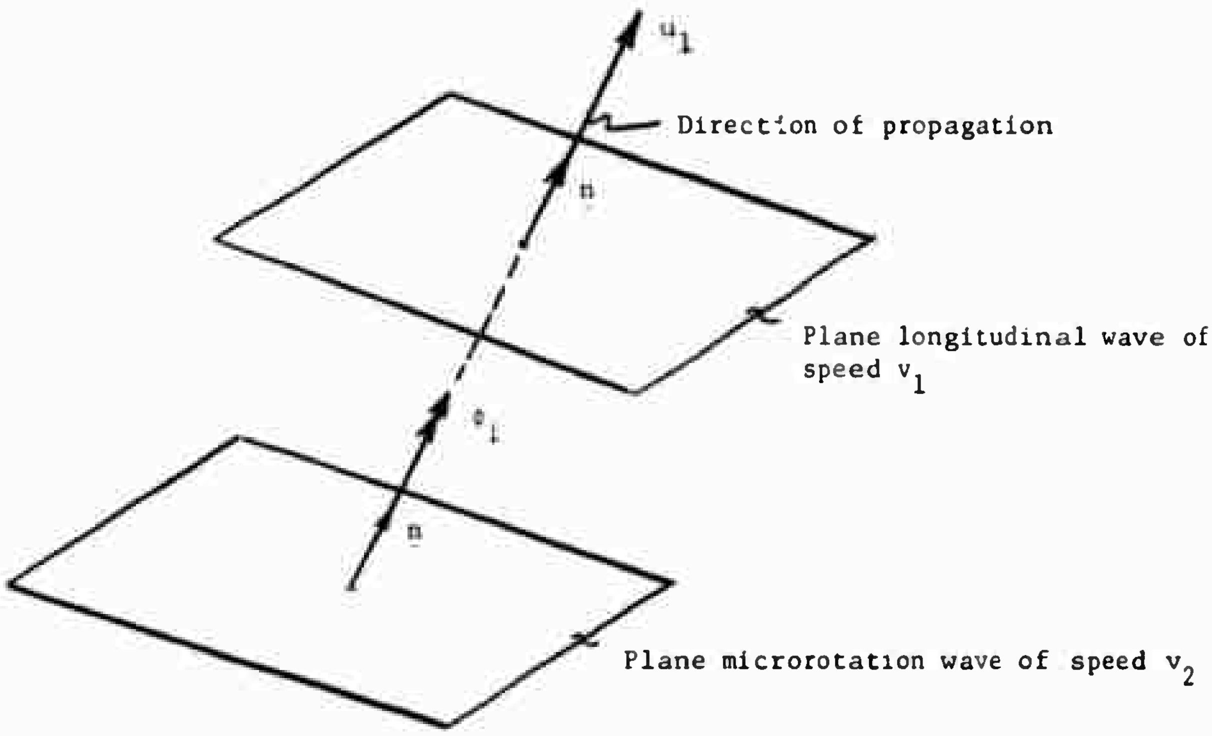

FIG. 24.1. LONGITUDINAL DISPLACEMENT AND MICROROTATION WAVES

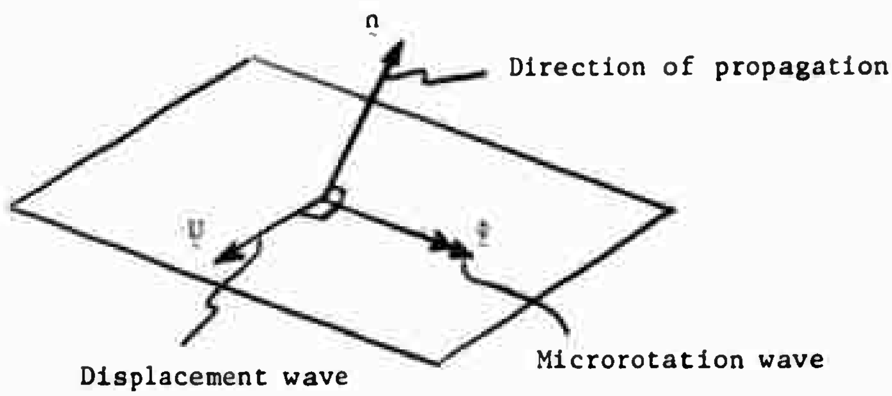

FIG. 24.2. COUPLED TRANSVERSE VECTOR WAVES PROPAGATING WITH SPEEDS v_3 AND v_4

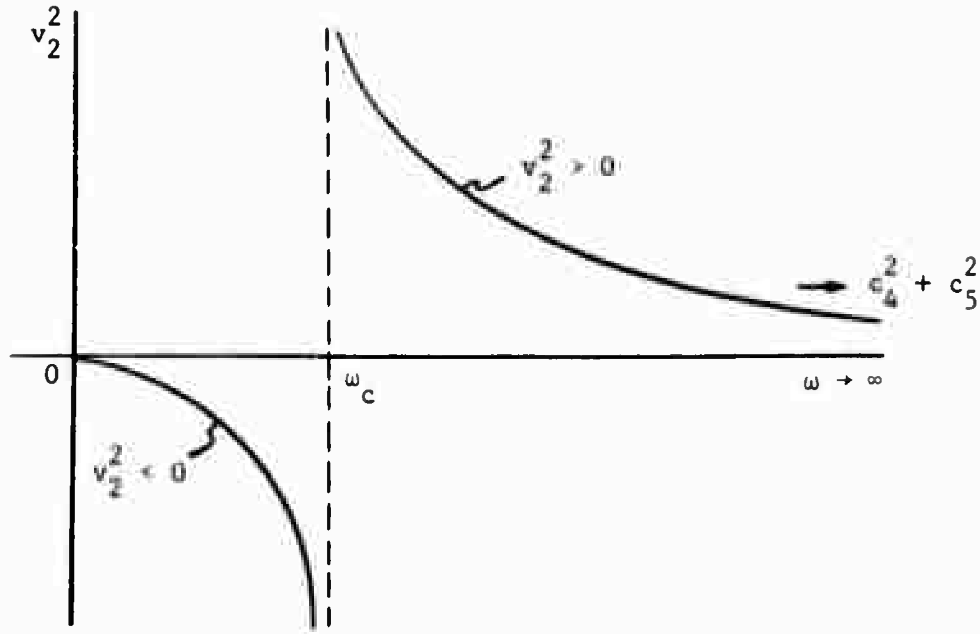

FIG. 24.3. SKETCH OF v_2^2 VERSUS ω

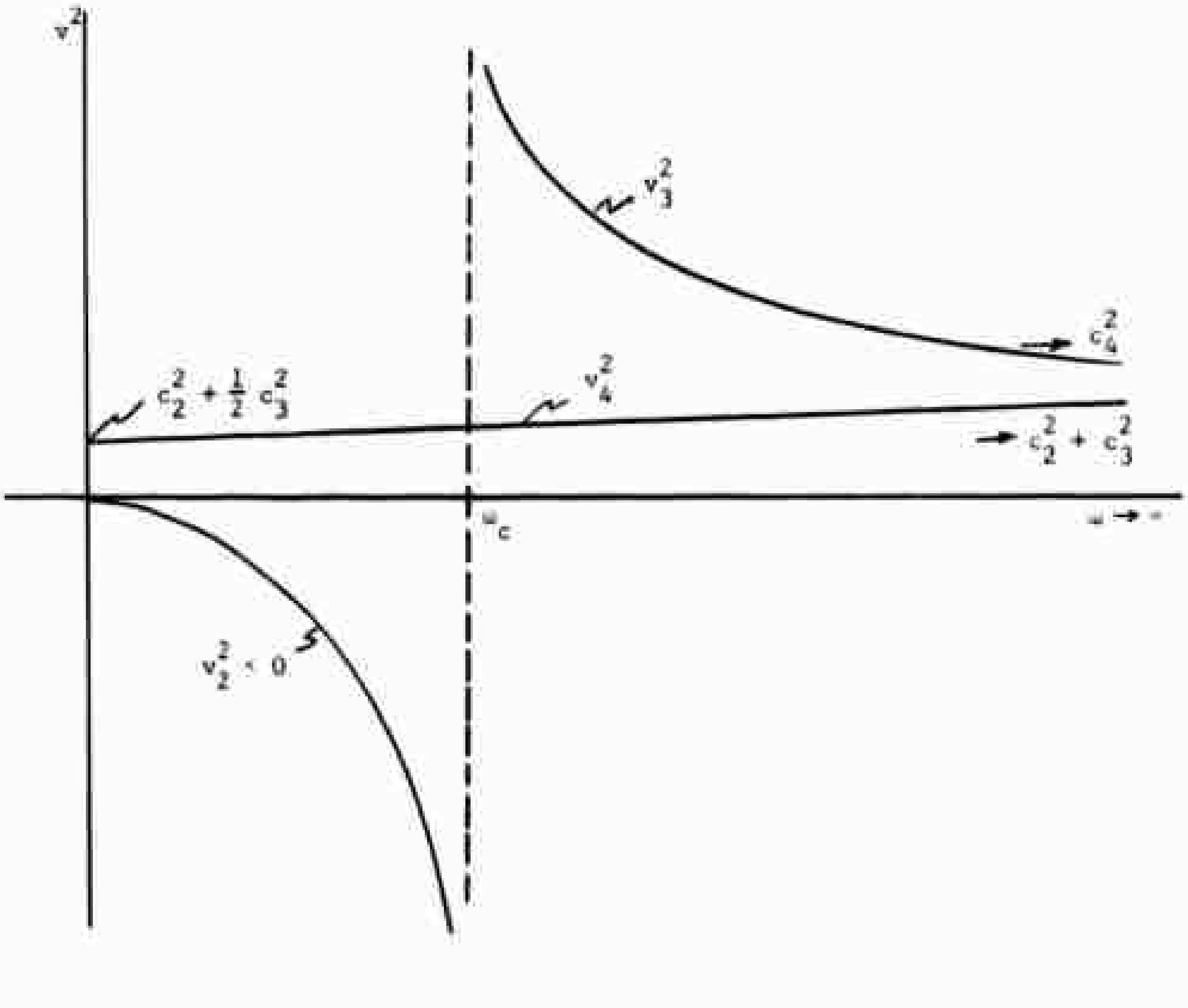

FIG. 24.4. SKETCH OF v_3^2 AND v_4^2 VERSUS ω

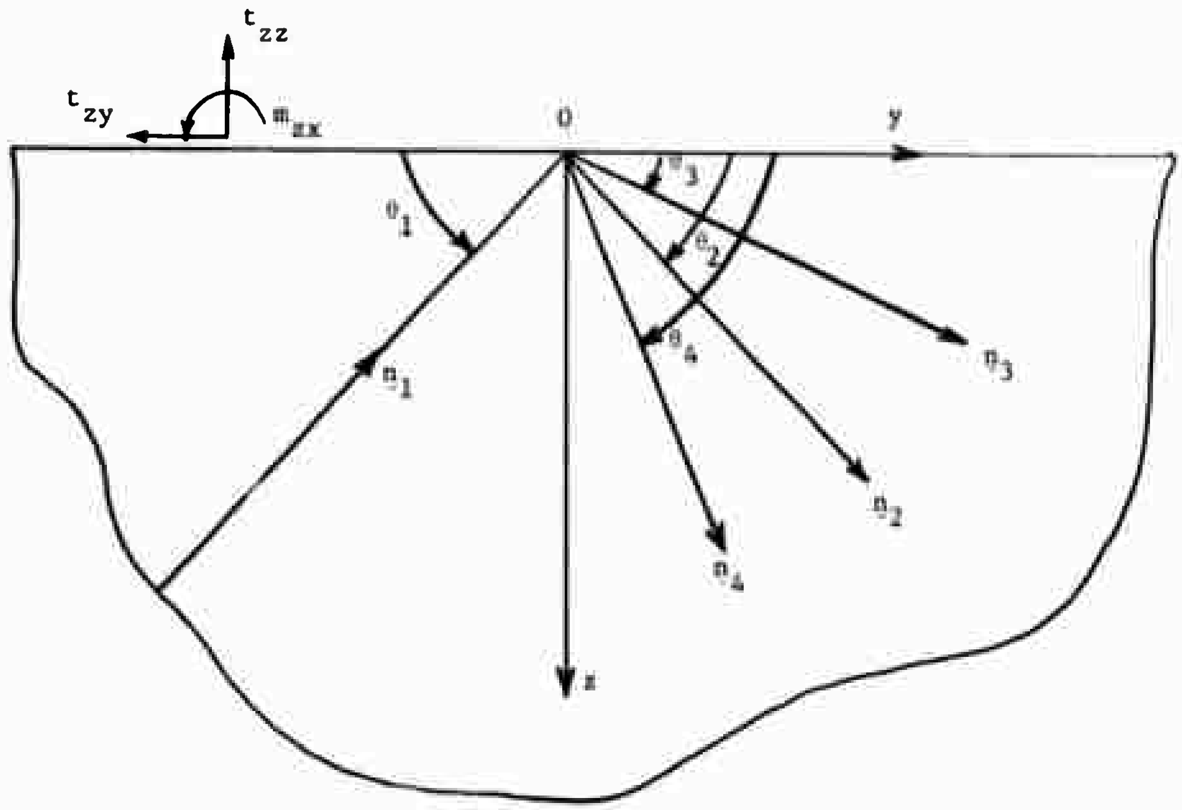

FIG. 25.1. REFLECTION OF A LONGITUDINAL DISPLACEMENT WAVE

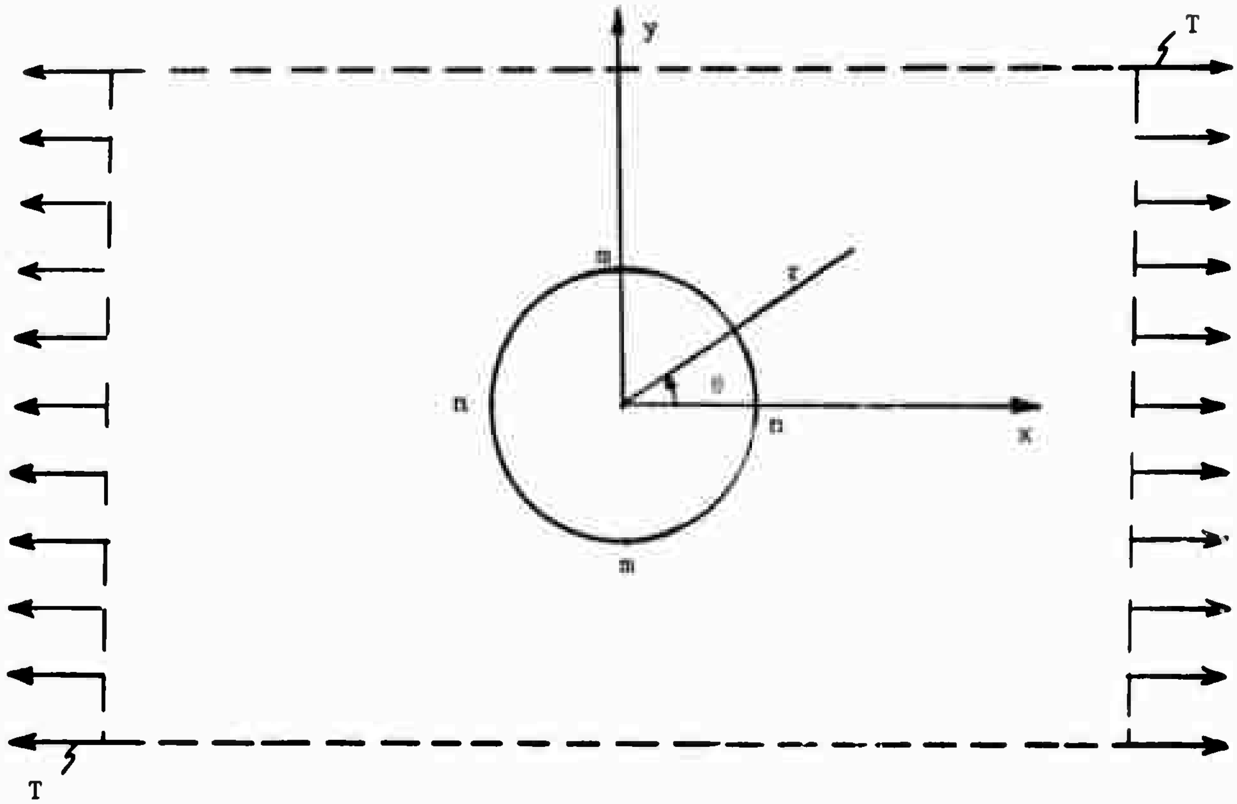

FIG. 27.1. CIRCULAR HOLE IN TENSION FIELD

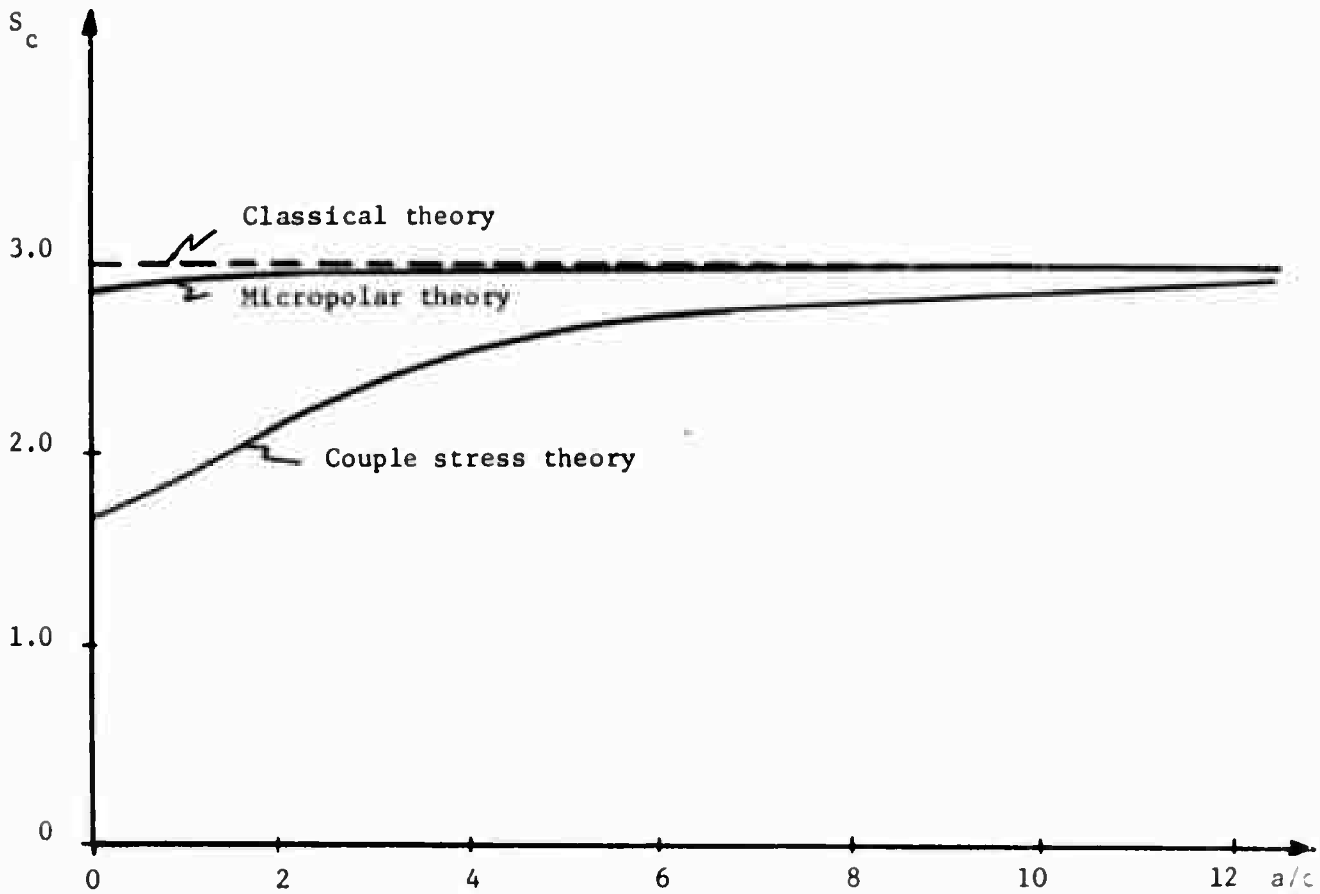

FIG. 27.2. STRESS CONCENTRATION FACTORS FOR $\frac{b}{c} = 0.20$, $\nu = 0$

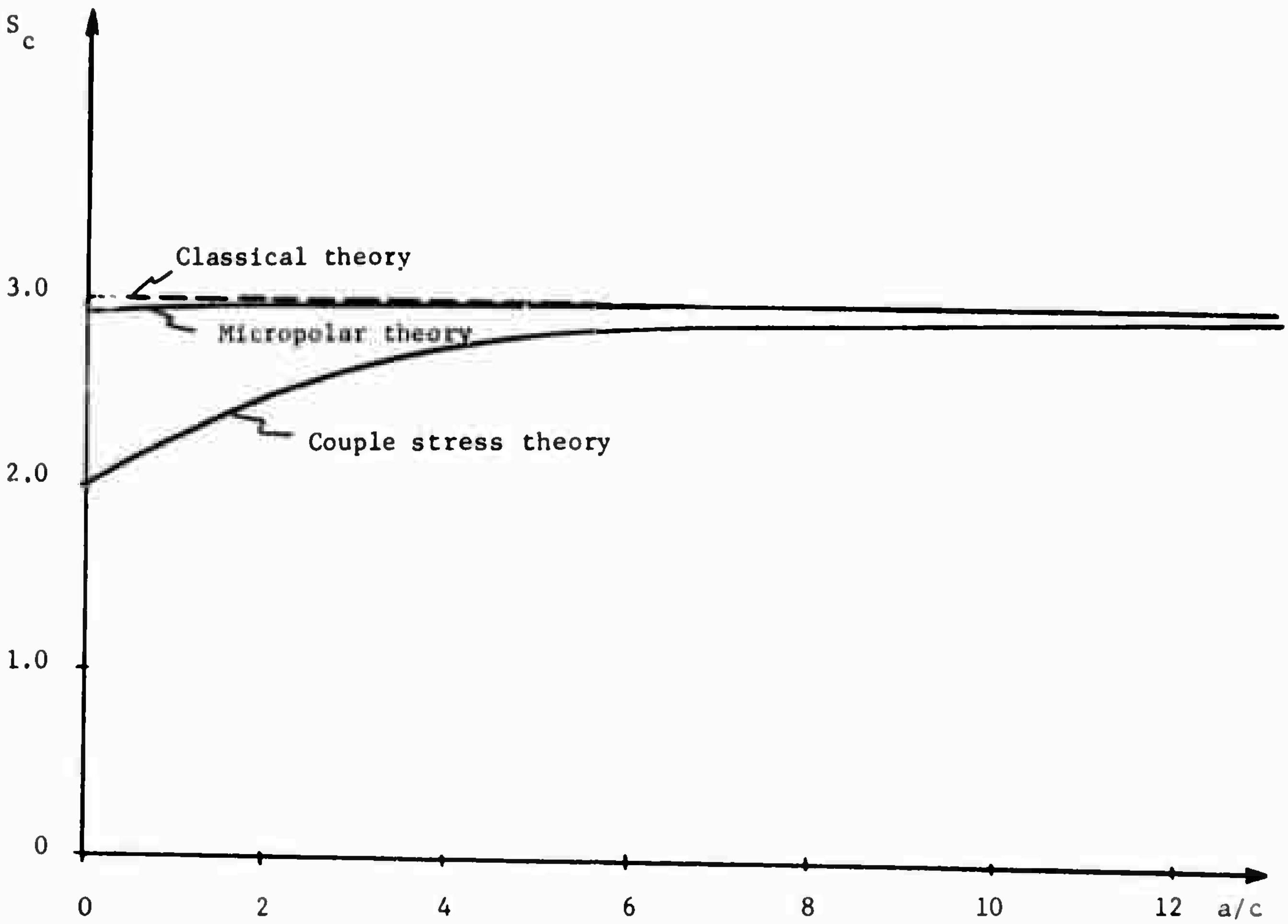

FIG. 27.3. STRESS CONCENTRATION FACTORS FOR $\frac{b}{c} = 0.20$, $\nu = 0.5$

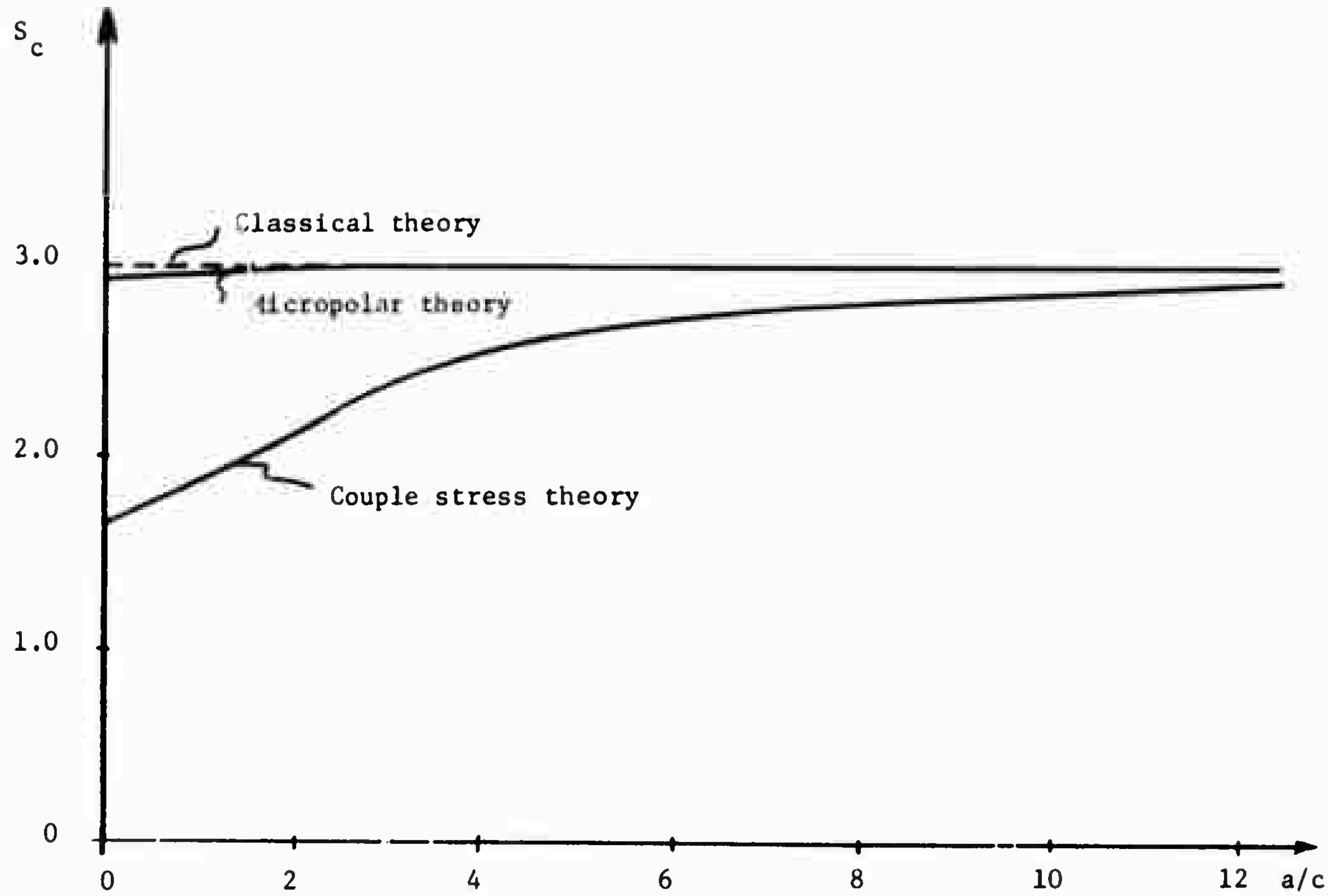

FIG. 27.4. STRESS CONCENTRATION FACTORS FOR $\frac{b}{c} = 0.10$, $\nu = 0$

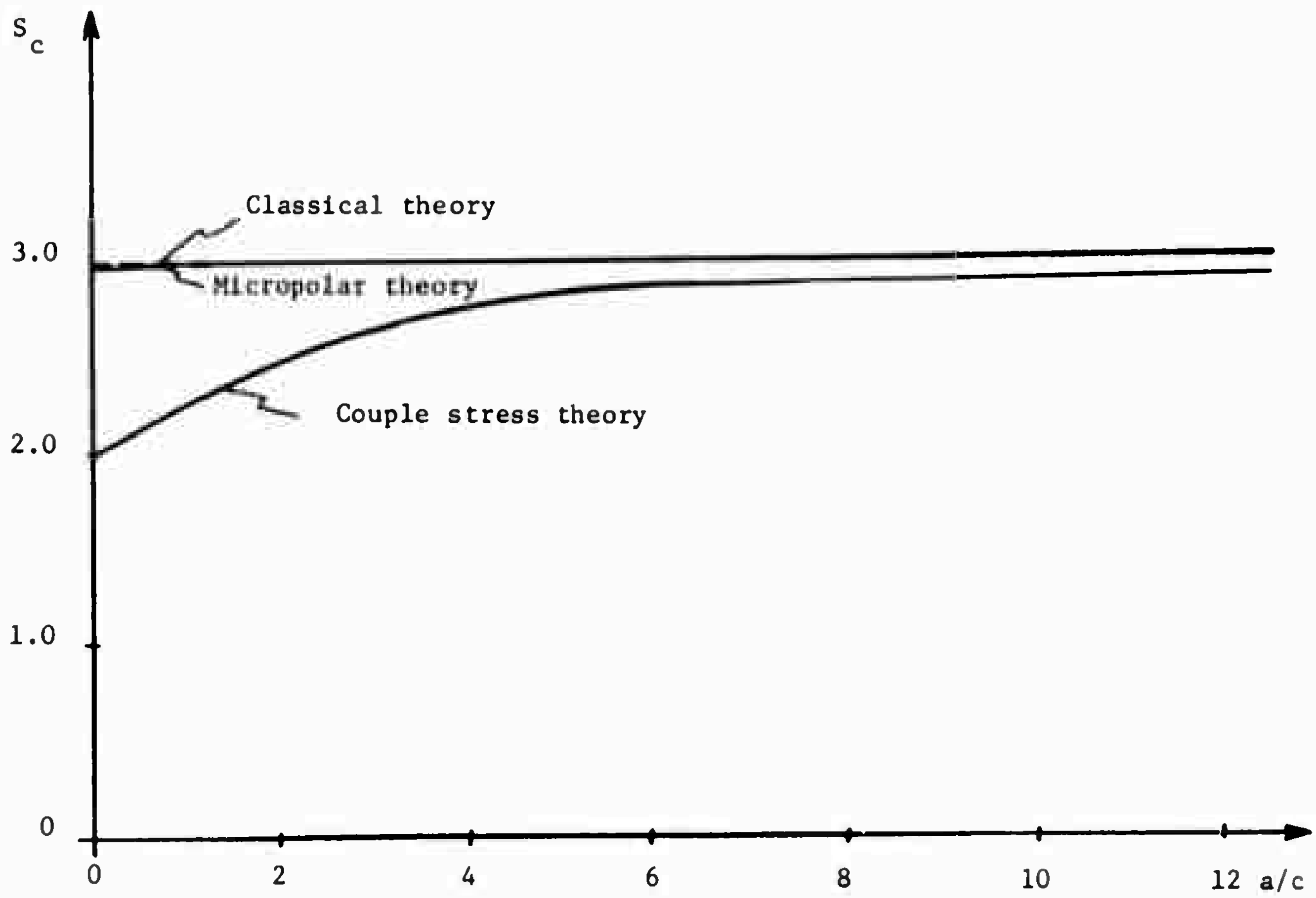

FIG. 27.5. STRESS CONCENTRATION FACTORS FOR $\frac{b}{c} = 0.10$, $\nu = 0.5$

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13 ABSTRACT

This article presents a self-contained account of the recent theory of micropolar elasticity. Micropolar elastic materials possess extra independent degrees of freedom for the local rotations different from the rotations of the classical elasticity. These materials respond to spin inertia and body and surface couples, and as a consequence they exhibit certain new static and dynamic effects, e.g., new types of waves and couple stresses.

Extensive discussions are presented on deformation, strain, microstrain, rotations, microrotations, kinematics, and balance laws. The thermodynamics of micropolar solids is formulated and the consequences of entropy and inequality are discussed. Field equations, boundary and initial conditions are obtained. The indeterminate couple stress theory is shown to result as a special case of the theory. Several static and dynamic problems are solved on the subjects of reflection of various types of micropolar waves, surface waves, stress concentration around a circular hole, and force and moment singularities in infinite solids.

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